Functions, Graphs and Conic Sections

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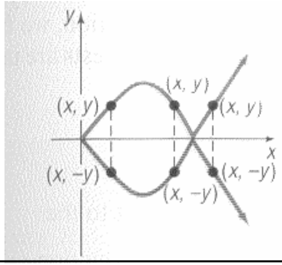
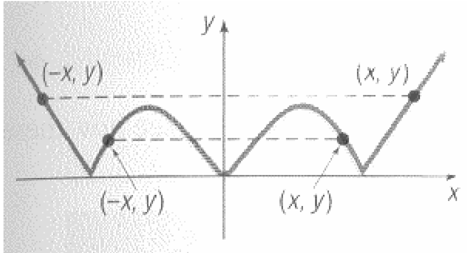
Symmetry with respect to x-axis

- A graph is said to be **Symmetric with respect to the x-axis** if, for every point (x, y) on the graph, the point (x, -y) is also on the graph.



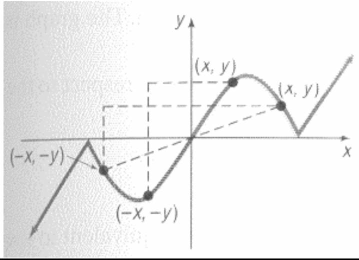
Symmetry with respect to y-axis

- A graph is said to be **Symmetric with respect to the y-axis** if, for every point (x, y) on the graph, the point (-x, y) is also on the graph.



Symmetry with respect to origin

- A graph is said to be **Symmetric with respect to the origin** if, for every point (x, y) on the graph, the point (-x, -y) is also on the graph.



Reflection

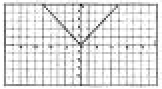
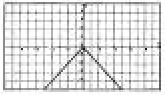
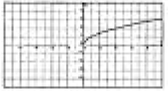
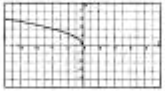
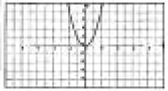
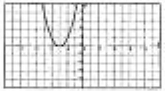
- The graph of y = –*f* (x) is a reflection in the x-axis of the graph of y = *f* (x).

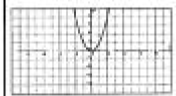
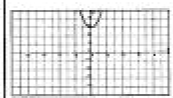
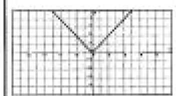
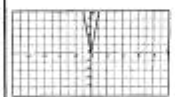
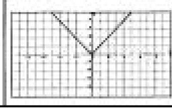
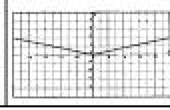
Translation

- Translating Upward or Downward
  - If k > 0, then the graph of y = f(x) + k is an **Upward Translation** of the graph of y = f(x) and the graph of y = f(x) – k is a **Downward Translation** of the graph of y = f(x)
- Translating to the Right and Left
  - If h > 0, then the graph y = f(x – h) is a **Translation to the Right** of the graph of y = f(x), and the graph of y = f(x + h) is a **Translation to the Left** of the graph of y = f(x)

Stretching and Shrinking

- If a > 1, then the graph of y = af(x) is obtained by **Stretching** the graph of y = f(x).
- If 0 < a < 1, then the graph of y = af(x) is obtained by **Shrinking** the graph of y = f(x).

Function	Description of change	Example	
		original	*transformed*
$y = -f(x)$	reflection across the *x*-axis	$y = \|x\|$ 	$y = -\|x\|$ 
$y = f(-x)$	reflection across the *y*-axis	$y = \sqrt{x}$ 	$y = \sqrt{-x}$ 
$y = f(x + h)$	translation of $-h$ in the *x*-direction	$y = x^2$ 	$y = (x + 3)^2$ 

Function	Description of change	Example *original*	Example *transformed*
$y = f(x) + h$	translation of $+h$ in the y-direction	$y = x^2$ 	$y = x^2 + 3$ 
$y = a \cdot f(x)$ $a > 1$	stretching by factor of a in the y-direction	$y = \lvert x \rvert$ 	$y = 5\lvert x \rvert$ 
$y = a \cdot f(x)$ $0 < a < 1$	shrinking by factor of a in the y-direction	$y = \lvert x \rvert$ 	$y = \frac{1}{3}\lvert x \rvert$ 

Example

Sketch the graph of $y = \sqrt{x + 2}$.

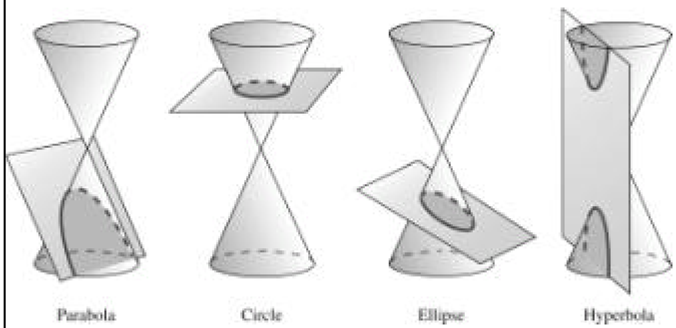
Example

Sketch the graph of $y = -2\,|\,x\,| + 3$.

Example

Sketch the graph of $y = x^2 + 2x + 3$.

Conic Sections

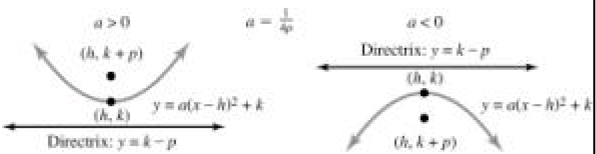


Parabola

- Given a line (the **Directrix**) and a point not on the line (the Focus), the set of all points in the plane that are equidistant from the point and the line is called a Parabola.

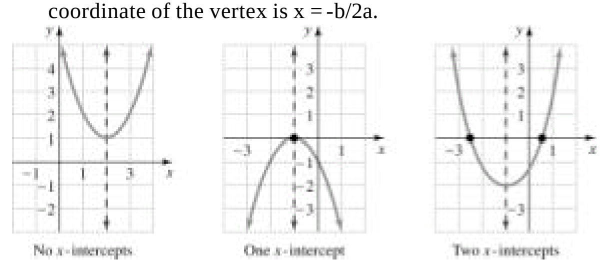
Characteristic of $y = (x - h)^2 + k$

- The graph of the equation $y = (x - h)^2 + k$ (for $a \neq 0$) is a parabola with vertex (h, k), focus (h, k + p), and directrix $y = k - p$, where $a = 1/4p$. If $a > 0$, the parabola open upward; if $a < 0$, the parabola opens downward.



Characteristic of $y = ax^2 + bx + c$

- The graph of $y = ax^2 + bx + c$ (for $a \neq 0$) is a parabola opening upward if a > o and downward if $a < 0$. The x-coordinate of the vertex is $x = -b/2a$.



Graphing a Parabola

Graphing the Parabola $y = ax^2 + bx + c$

To graph the parabola $y = ax^2 + bx + c$, use the following facts:

1. The parabola opens upward if $a > 0$ and opens downward if $a < 0$.
2. The x-coordinate of the vertex is $\frac{-b}{2a}$.
3. The graph is symmetric about the vertical line $x = \frac{-b}{2a}$.
4. The y-intercept is $(0, c)$.
5. The x-intercepts are found by solving $ax^2 + bx + c = 0$.

Example

- Graph the parabola $y = 3x^2 + 6x + 1$

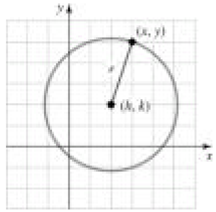
Circle

- A **Circle** is the set of all points in a plane that lie a fixed distance from a given point in the plane. The fixed distance is called the radius, and the given points is called the **Center**.

Standard Equation for a Circle

- The graph of the equation
  - $(x - h)^2 + (y - k)^2 = r^2$

  with $r > 0$, is a circle with center (h, k) and radius r.



Example

- Find the center and radius of the circle given by $2x^2 - 3x + 2y^2 + 7y = -5$

Intersection of a line and a circle

- Find the intersection of the following equation:
  - $(x - 3)^2 + (y + 1)^2 = 9$
  - $y = x - 1$

Ellipse

- An Ellipse is the set of all points in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is call a **Focus**.

Equation of an Ellipse centered at Origin

- An Ellipse centered at (0, 0) with focus at (c, 0) and constant sum 2a has equation:

  $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

  where a, b and c are positive real numbers with $c^2 = a^2 - b^2$.

Example

Find the x- and y-intercepts for the ellipse and sketch its graph.

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Equation of an Ellipse centered at (h, k)

- An ellipse centered at (h, k) has equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where a and b are positive real numbers.

Example

Sketch the graph of the ellipse:

$$\frac{(x - 1)^2}{9} + \frac{(y + 2)^2}{4} = 1$$

Hyperbola

- A hyperbola is the set of all points in the plane such that the difference at their distances from two fixed points (focus) is constant.

Equation of a Hyperbola centered at Origin

- An Hyperbola centered at (0, 0) with focus at (c, 0) and (-c, 0) and constant difference 2a has equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a, b and c are positive real numbers with $c^2 = a^2 + b^2$.

Graphing Hyperbola

Graphing a Hyperbola Centered at the Origin, Opening Left and Right

To graph the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

1. Locate the x-intercepts at $(a, 0)$ and $(-a, 0)$.
2. Draw the fundamental rectangle through $(\pm a, 0)$ and $(0, \pm b)$.
3. Draw the extended diagonals of the rectangle to use as asymptotes.
4. Draw the hyperbola to the left and right approaching the asymptotes.

Graphing Hyperbola

Graphing a Hyperbola Centered at the Origin, Opening Up and Down

To graph the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$:

1. Locate the y-intercepts at $(0, b)$ and $(0, -b)$.
2. Draw the fundamental rectangle through $(0, \pm b)$ and $(\pm a, 0)$.
3. Draw the extended diagonals of the rectangle to use as asymptotes.
4. Draw the hyperbola opening up and down approaching the asymptotes.

Example

Sketch the graph of $\frac{x^2}{36} - \frac{y^2}{9} = 1$, and find the equations of its asymptotes.

Sketch the graph of the hyperbola $4x^2 - y^2 = 4$.