Rational Exponents and Radicals

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n^{th} Roots

- The number b is an n^{th} root of a if
  - $b^n = a$
- If n is a positive integer, then
  - $0^{1/n} = 0$

Principle n^{th} root

- Exponent 1/n when n is even
  - If n is a positive even integer and a is a positive real number, then $a^{1/n}$ denotes the positive real n^{th} root of a and is called **principle n^{th} root of a**.
- Exponent 1/n when n is odd
  - If n is a positive odd integer and a is any real number, then $a^{1/n}$ denotes the real n^{th} root of a.

Examples:

- Evaluate each expressions:
  - $4^{1/2}$
  - $16^{1/4}$
  - $-81^{1/4}$
  - $8^{1/3}$
  - $(-27)^{1/3}$
  - $-32^{1/5}$
  - $0^{1/3}$

Rational Exponents

- If m and n are positive integers, then
  - $\mathbf{a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}}$,

  provided that $a^{1/n}$ is defined.

- If m and n are positive integers, then
  - $a^{-m/n} = \dfrac{1}{a^{m/n}}$,

  provided that $a^{1/n}$ is defined and non-zero.

Important Rules

Rules for Rational Exponents

The following rules hold for any nonzero real numbers a and b and rational numbers r and s for which the expressions represent real numbers.

1. $a^r a^s = a^{r+s}$	Product rule
2. $\dfrac{a^r}{a^s} = a^{r-s}$	Quotient rule
3. $(a^r)^s = a^{rs}$	Power of a power rule
4. $(ab)^r = a^r b^r$	Power of a product rule
5. $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}$	Power of a quotient rule

Simplifying Expressions involving Variables

- When simplifying expressions involving rational exponents and variables, we must be careful to write equivalent expressions.

Square Root of x^2

$(x^2)^{1/2} = |x|$ for any real number x.

Using absolute value symbols with exponents

- Simplify each expression.
  - $(x^8 y^4)^{\frac{1}{4}}$
  - $\left(\dfrac{x^9}{8}\right)^{\frac{1}{3}}$

Expressions involving variables with rational exponents

- Use the rules of exponents to simplify the following and give the answer in positive exponents. Assume that all variables represent positive real number.

$a)\quad x^{\frac{2}{3}}x^{\frac{4}{3}}$ $\qquad c)\quad (x^{\frac{1}{2}}y^{-3})^{\frac{1}{2}}$

$b)\quad \dfrac{a^{\frac{1}{2}}}{a^{\frac{1}{4}}}$ $\qquad d)\quad \left(\dfrac{x^2}{y^{\frac{1}{3}}}\right)^{-\frac{1}{2}}$

Radical Notation

Radicals

If n is a positive integer and a is a real number for which $a^{1/n}$ is defined, then the expression $\sqrt[n]{a}$ is called a **radical,** and

$$\sqrt[n]{a} = a^{1/n}.$$

Product and Quotient Rules for Radicals

Product Rule for Radicals

The nth root of a product is equal to the product of the nth roots. In symbols,

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b},$$

provided that all of the expressions represent real numbers.

Quotient Rule for Radicals

The nth root of a quotient is equal to the quotient of the nth roots. In symbols,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}},$$

provided that all of the expressions represent real numbers and $b \neq 0$.

Simplified Radical Form for Radicals of index n

- A radical expression of index n is in simplified radical form if it has
  - No perfect nth power as factors of the radicand.
  - No fractions inside the radical
  - No radicals in the denominator.
- Examples
  - Simplify $\dfrac{\sqrt[4]{2x^{11}} \cdot \sqrt[4]{16x^{10}}}{\sqrt[4]{2x^3}}$.

Adding and Subtracting Radicals

Simplify $\sqrt{98} - \sqrt{50}$.

Multiplying Radicals

Simplify $\left(3\sqrt{2} - \sqrt{5}\right)\left(3\sqrt{2} + 2\sqrt{5}\right)$.

Conjugates

Simplify $\left(3 + \sqrt{x+2}\right)^2$.

Dividing Radicals

Simplify the expression $\dfrac{3+\sqrt{x}}{2-\sqrt{x}}$.

Odd-Root and Even-Root Property

- Odd-Root Property

  If n is an odd positive integer,

  $$x^n = k \quad \text{is equivalent to} \quad x = \sqrt[n]{k}$$

  for any real number k.

- Even-Root Property

  Suppose n is a positive even integer.

  If $k > 0$, then $x^n = k$ is equivalent to $x = \pm\sqrt[n]{k}$.

  If $k = 0$, then $x^n = k$ is equivalent to $x = 0$.

  If $k < 0$, then $x^n = k$ has no real solution.

Solving Equations with Exponents and Radicals

Strategy for Solving Equations with Exponents and Radicals

1. In raising each side of an equation to an even power, we can create an equation that gives extraneous solutions. We must check all possible solutions in the original equation.
2. When applying the even-root property, remember that there is a positive and a negative even root for any positive real number.
3. For equations with rational exponents, raise each side to a positive or negative integral power first, then apply the even- or odd-root property. (Positive fraction—raise to a positive power; negative fraction—raise to a negative power.)

Example

Solve the equation $(2x-3)^2 = 5$.

Solve the equation $\sqrt{x+7} = x+5$.

Solve the equation $\sqrt{2x-1} + \sqrt{x-1} = 5$.