Rational Expressions

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Definition of Rational Expressions

- A **Rational Expressions** is the ratio of two polynomials with the denominator not equal to zero.
- The domain of a rational expressions is the set of all real numbers that can be used in place of the variable.

Basic Principle of Rational Numbers

Basic Principle of Rational Numbers

If $\frac{a}{b}$ is a rational number and c is a nonzero real number, then

$$\frac{a}{b} = \frac{ac}{bc}.$$

Example

- Reducing $\dfrac{-2a^7b}{a^2b^3}$
- Reducing $\dfrac{2a^3-16}{16-4a^2}$

Reducing Rational Expressions

Strategy for Reducing Rational Expressions

1. All reducing is done by dividing out common factors.
2. Factor the numerator and denominator completely to see the common factors.
3. Use the quotient rule to reduce a ratio of two monomials involving exponents.
4. We may have to factor out a common factor with a negative sign to get identical factors in the numerator and denominator.

Example

- Mercedes Benz spent $700 million to develop its new 1999 M class SUV, which will sell for around $40,000. If the cost of manufacturing the SUV is $30,000 each, then what rational function gives the average cost of developing and manufacturing x vehicles? Compare the average cost per vehicle for manufacturing levels of 10,000 vehicles and 100,000 vehicles.

Multiplication and Division

Multiplication of Rational Numbers

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

Division of Rational Numbers

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers with $\frac{c}{d} \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

Example

- Reducing $\frac{a^3 - b^3}{b - a} \cdot \frac{6}{2a^2 + 2ab + 2b^2}$
- Reducing $\frac{\frac{x^2 - 4}{2}}{\frac{x - 2}{3}}$

Addition and Subtraction

Addition and Subtraction of Rational Numbers

If $b \neq 0$, then

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \quad \text{and} \quad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}.$$

- Example $\dfrac{x^2+4x+7}{x^2-1} - \dfrac{x^2-2x+1}{x^2-1}$

Least Common Denominator

- To add fractions with denominators that are not identical, Least Common Denominator (LCD) is the basic principle to build up the denominators

Strategy for Finding the LCM for Polynomials

1. Factor each polynomial completely. Use exponents to express repeated factors.
2. Write the product of all of the different factors that appear in the polynomials.
3. For each factor, use the highest power of that factor in any of the polynomials.

Fractions Operations

Adding or Subtracting Simple Fractions

If $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}.$$

Example

- Simplify $\dfrac{1}{x^2-1} + \dfrac{2}{x^2+x}$
- Simplify $\dfrac{5}{a-2} - \dfrac{3}{2-a}$
- Simplify $\dfrac{\dfrac{x+2}{x^2-9}}{\dfrac{x}{x^2-6x+9} + \dfrac{4}{x-3}}$

Solving Equations involving Rational Expressions

- To solve an equation that contains rational expressions:
  - Find the LCD
  - Multiply both sides of the equations by LCD
  - Solve the resulting polynomial equation
  - Check all solutions in the original equation. Remove any extraneous solutions from the solution set.

An Equation with Rational Expressions

Solve $\frac{1}{x} + \frac{2}{3x} = \frac{1}{5}$.

An Equation with Two Solutions

Solve $\frac{200}{x} + \frac{300}{x + 20} = 10$.

An Equation with Extraneous Root

Solve $x + 2 + \frac{x}{x - 2} = \frac{2}{x - 2}$.

Proportions

- The equation ad = bc says that product of extremes is equal to the product of the means. When solving a proportion, we can omit multiplication by the LCD and just remember the result, ad = bc, as the extremes-means property.

Extremes-Means Property

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

A Proportion with One Solution

Solve $\frac{20}{x} = \frac{30}{x + 20}$.

A Proportion with Two Solutions

Solve $\frac{2}{x} = \frac{x + 3}{5}$.

Application Examples

- Michele drove her empty rig 300 miles to Salina to pick up a load of cattle. When her rig was fully loaded, her average speed was 10 miles per hour less than when the rig was empty. If the return trip took her 1 hour longer, then what was her average speed with the rig empty?