Polynomial Functions

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Polynomials

- A **Polynomial** is defined as single term or a sum of a finite number of terms.
  - $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$
- **Term** is a single number of the product of a number or one or more variables raised to whole number powers.
- The number preceding the variable in each term is called the **Coefficient**.
- A number is referred to as a **Constant**.

Operation of Polynomials

Addition and Subtraction of Polynomials

To add two polynomials, add the like terms.
To subtract two polynomials, subtract the like terms.

Multiplication of Polynomials

To multiply polynomials, multiply each term of the first polynomial by each term of the second polynomial and then combine like terms.

Example

- Find the sums
  - $(x^2 - 5x - 7) + (7x^2 - 4x + 10)$
  - $(3x^3 - 5x^2 - 7) + (4x^2 - 2x + 3)$
- Find the differences
  - $(x^2 - 7x - 2) - (5x^2 + 6x - 4)$
  - $(6y^3z - 5yz + 7) - (4y^2z - 3yz - 9)$

Example

- Find the product of $(x + 2)(x^2 + 3x - 5)$
- Multiply $(3x^2 + 4)(x^2 - 7x + 2)(x + 3)$

Multiplying Binomials (FOIL)

Consider how we find the product of two binomials $x + 3$ and $x + 5$ using the distributive property twice:

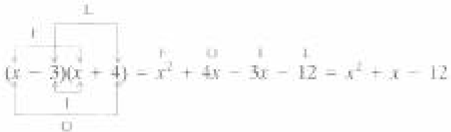
$$\begin{aligned}(x+3)(x+5) &= (x+3)x + (x+3)5 && \text{Distributive property}\\ &= x^2 + 3x + 5x + 15 && \text{Distributive property}\\ &= x^2 + 8x + 15 && \text{Combine like terms.}\end{aligned}$$

There are four terms in the product. The term x^2 is the product of the first term of each binomial. The term $5x$ is the product of the two outer terms, 5 and x. The term $3x$ is the product of the two inner terms, 3 and x. The term 15 is the product of the last two terms in each binomial, 3 and 5. It may be helpful to connect the terms multiplied by lines.

Multiplying Binomials (FOIL)

So instead of writing out all of the steps in using the distributive property, we can get the result by finding the products of the first, outer, inner, and last terms. This method is called the **FOIL method.**

For example, let's apply FOIL to the product $(x - 3)(x + 4)$:



If the outer and inner products are like terms, you can save a step by writing down only their sum.

Square of Binomial

- Rule for the Square of a Sum
  - $(a + b)^2 = a^2 + 2ab + b^2$
- Rule for the Square of a Difference
  - $(a - b)^2 = a^2 - 2ab + b^2$
- Rules of a Sum and a Difference
  - $(a + b)(a - b) = a^2 - b^2$

Example

- Multiply (3x + 2y)(3x – 2y)

- Multiply (2x + 4)(2x – 3)

Division of Polynomials

- Divide $4x^3 - x - 9$ by $2x - 3$.

Synthetic Division

- When dividing a polynomial by a binomial of form x – c, we can use Synthetic Division to speed up the process. For Synthetic Division, we write only the essential parts of ordinary division.

- Example
  - Divide $x^3 - 5x^2 + 4x - 3$ by $x - 2$.

Strategy for using Synthetic Division

Strategy for Using Synthetic Division

1. List the coefficients of the polynomial (the dividend).
2. Be sure to include zeros for any missing terms in the dividend.
3. For dividing by $x - c$, place c to the left.
4. Bring the first coefficient down.
5. Multiply by c and add for each column.
6. Read $Q(x)$ and R from the bottom row.

Factoring Polynomials

- Factoring out the Great Common Factor
- Factoring out the Opposite of the Great Common Factor
- Factoring the Difference of Two Squares
- Factoring Perfect Square Trinomial
- Factoring a Different or a Sum of Two Cubes
- Factoring a Polynomial Completely
- Factoring by Substitution

Factoring out the Great Common Factor

Strategy for Finding the Greatest Common Factor (GCF)

1. Factor each term completely.
2. Write a product using each factor that is common to all of the terms.
3. On each of these factors, use an exponent equal to the smallest exponent that appears on that factor in any of the terms.

- Example:
  - Factorize $18x^3 - 6x^2$.

Factoring Perfect Square Trinomials

- The trinomial that results from squaring a binomials is call **Perfect Square Trinomial**.
  - $a^2 + 2ab + b^2 = (a + b)^2$
  - $a^2 - 2ab + b^2 = (a - b)^2$

Strategy for Identifying Perfect Square Trinomials

A trinomial is a perfect square trinomial if

1. the first and last terms are of the form a^2 and b^2,
2. the middle term is 2 or -2 times the product of a and b.

Example

- Factorize $9y^2 - 64$.

Factoring a Difference or a Sum of Two Cube

- $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
- $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

- Example:
  - Factorize $27x^3 + 64$

Factoring $ax^2 + bx + c$

- Strategy for factoring $ax^2 + bx + c$ by the ac Method:
  - To factor the trinomial $ax^2 + bx + c$
    - Find two integers that have a product equal to *ac* and a sum equal to *b*.
    - Replace *bx* by two terms using the two new integers as coefficients.
    - Then factor the resulting four-term polynomial by grouping

Factoring Strategy

Strategy for Factoring Polynomials

1. If there are any common factors, factor them out first.
2. When factoring a binomial, look for the special cases: difference of two squares, difference of two cubes, and sum of two cubes. Remember that a sum of two squares $a^2 + b^2$ is prime.
3. When factoring a trinomial, check to see whether it is a perfect square trinomial.
4. When factoring a trinomial that is not a perfect square, use grouping or trial and error.
5. When factoring a polynomial of high degree, use substitution to get a polynomial of degree 2 or 3, or use trial and error.
6. If the polynomial has four terms, try factoring by grouping.

Example

- Factorize $3x^8 - 243$ completely

Solving Equations by Factoring

Strategy for Solving Equations by Factoring

1. Write the equation with 0 on the right-hand side.
2. Factor the left-hand side.
3. Use the zero factor property to get simpler equations. (Set each factor equal to 0.)
4. Solve the simpler equations.
5. Check the answers in the original equation.

Example

- Solve $x(x + 2) = 3$.

- Solving the equation $2x^3 - x^2 - 8x + 4 = 0$.