Algebra Equations and Inequalities

Peter Lo

What is Equation?

- Equation is a statement that has two expressions are equal.
  - Examples:
    - $X + 2 = 9$
    - $11y = 5y + 6y$
    - $X^2 - 2x - 1 = 0$
- Two or more equations have precisely the same solutions are called **Equivalent Equations**.

Identity

- An equation satisfied by every number that is a meaningful replacement for the variable is called an **Identity**.

Solve the equation $2(x - 1) + 4 = 4(1 + x) - (2x + 2)$.

Conditional Equation

- Equations satisfied by some numbers but not by others are called **Conditional Equation**.

Solve the equation $\frac{x+2}{5} - 4x = \frac{8}{5} - \frac{x+9}{2}$.

Contradiction

- The Equation is neither an Identity nor a Conditional Equation is called a **Contradiction**.

Solve the equation $\frac{x-1}{3} + 4x = \frac{3}{2} + \frac{13x-2}{3}$.

Equation Solving

- **Solve an equation** means to find all number that make the equation a true statement.
- Such numbers called **Solutions** or **Roots** of the equation.
- A number that is a solution of an equation is said to **Satisfy** the equation, and the solutions of an equation makeup its **Solution Set**. Equations with the same solutions set are called **Equivalent Equation**.

Procedures that Result in Equivalent Equations

- Interchange the two side of the equations.
  - Replace 3 = x by x = 3.
- Simplify the sides of the equation by combining like terms, eliminating parentheses, and so on.
  - Replace (x + 2) + 6 = 2x + (x + 1)
    by x + 8 = 3x + 1
- Add or subtract the same expression on both sides of the equation.
  - Replace 3x – 5 = 4
    by (3x – 5) + 5 = 4 + 5

Procedures that Result in Equivalent Equations

- Multiply or divide sides of the equation by the same non-zero expression:

Replace $\frac{3x}{x-1} = \frac{6}{x-1}$ where $x \neq 1$

by $\frac{3x}{x-1} \cdot (x-1) = \frac{6}{x-1} \cdot (x-1)$

- If one side of the equation is 0 and the other side can be factored, then we may use the **Zero-Product Property** and set each factor equal to 0.
  - Replace x(x – 3) = 0
    by x = 0 or x – 3 = 0.

Steps for Solving an Equation

- List any restrictions on the domain of the variable
- Simplify the equation by replacing the original equation by a succession of equations following the procedures listed earlier.
- If the result of above is a product of factors equal to 0, use the Zero-Product Property and set each factor equal to 0.
- Check your solution.

Linear Equation

- An Linear Equation in one variable is equivalent to an equation of this form **ax + b** where a and b are real numbers and a ≠ 0.
- A linear equation is called a **First-Degree Equation** because the left side is a polynomial in x of degree 1.

Example 1

Solve the equation: $\frac{1}{2}(x + 5) - 4 = \frac{1}{3}(2x - 1)$

Example 2

Solve the equation: $\frac{3x}{x - 1} + 2 = \frac{3}{x - 1}$

Quadratic Equation

- Any equation of the form $ax^2 + bx + c = 0$ where a, b and c are real numbers and $a \neq 0$ is called a **Quadratic Equation**.
- In general, there are four methods to solve a quadratic equation.
  - Factorization
  - Completing square
  - Quadratic formula
  - Graphical method

Factorization

- It makes use the fact that $a \cdot b = 0$ implies $a = 0$ or $b = 0$. The quadratic expression is factorized into two linear factors and hence the solution of the equation is obtained by solving two linear equations.

Example 3

- Solve $12x^2 - 23x + 10 = 0$.

Completing Square

- This method is based on the fact that a quadratic equation $x^2 + px + q$ may be put into the form $(x + l)^2 - m^2 = 0$ and hence the solutions of the equation are $x = -l + m$. In order to put in the form $(x + l)^2 - m^2 = 0$, we have to make use the following identities.
  - $(x + y)^2 = x^2 + 2xy + y^2$
  - $(x - y)^2 = x^2 - 2xy + y^2$

Completing Square

Consider the equation $x^2 + px + q = 0$

$$x^2 + px + \left(\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 + q = 0$$

$$\left(x + \frac{p}{2}\right)^2 - \left[\left(\frac{p}{2}\right)^2 + q\right] = 0$$

By comparing with the equation $(x + l)^2 - m^2 = 0$, we have

$$l = \frac{p}{2} \quad , \quad m^2 = \left(\frac{p}{2}\right)^2 - q$$

Example 4

- Solve $2x^2 + 6x = -3$ by using completing square method.

Quadratic Equation Formula

- It is a tedious work to find the solution of a quadratic equation by the factorization or completing square method. So we are going to derive a general solution formula for the equation $ax^2 + bx + c = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof of the Equation

Proof: $ax^2 + bx + c = 0$

$$ax^2 + bx = -c$$

$$x^2 + \left(\frac{b}{a}\right)x = -\frac{c}{a}$$

$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 5

- Solve $3x^2 - 6x - 14 = 0$ by formula and give the answer in surd form.

Trips for Solving Quadratic Equation

- A general rule for solving quadratics equation is:
  - If the factors are equally seen, use the factorization.
  - Otherwise, use the quadratic formula; it always works!

Characteristic of the roots of Quadratic Equation

- The character of the roots of $ax^2 + bx + c = 0$ is determined by $b^2 - 4ac$, which is called Discriminant of the equation and is often denoted by the symbol Δ, i.e. $\Delta = b^2 - 4ac$.

If a, b and c are rational numbers, and
(1) if $\Delta > 0$, the two roots are real and unequal.
a) if Δ is a perfect square, the roots are rational.
b) if Δ is not a perfect square, the roots are irrational.
(2) if $\Delta = 0$, the two roots are rational and equal to $-\frac{b}{2a}$.
(3) if $\Delta < 0$, the two roots are unreal and unequal.

Example 6

- Find the value of k so that the equation $6x^2 + 5x + k = 0$ having equal roots.

- What is the nature of the roots of $2x^2 - 3x + 4 = 0$?

Sum and Product of the Roots of a Quadratic Equation

If α and β are the roots of the equation $ax^2 + bx + c = 0$, then

$ax^2 + bx + c = 0$

$(x - \alpha)(x - \beta) = 0$

$x^2 - \alpha x - \beta x + \alpha\beta = 0$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Equating the coefficients, $-a(\alpha + \beta) = b$ and $a(\alpha\beta) = c$

$$\text{Sum of roots} = \alpha + \beta = -\frac{b}{a}$$

$$\text{Product of roots} = \alpha\beta = \frac{c}{a}$$

Therfore, a quadratic equation $ax^2 + bx + c = 0$ can be written as

$$x^2 - (\text{Sum of roots})x + (\text{Product of roots}) = 0$$

Radical Equations

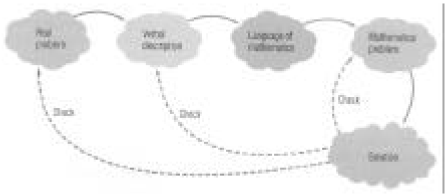
- When the variable in an equation occurs in a radical such as square root, cube root, and so on, the equation is called **Radical Equation**.

Example 7

Find the real solutions of the equation: $\sqrt{x - 1} = x - 7$

Business Application of Equations

- Translate Verbal Descriptions into Mathematics Expressions
- Set up Applied Problems
- Solve Interest Problems
- Solve Mixture Problems
- Solve Uniform Motion Problems
- Solve Constant Rate Job Problems



Business Concept

- When the regular price of merchandise is reduce, the amount of reduction is called **Markdown**.

  Sale price = regular price − markdown.

- Usually, the markdown is expressed as percent of the regular price.

  Markdown = percent of markdown · regular price.

Strategy for Business Problem Solving

1. Read the problem several times and analyze the facts given. What information is given? What are you asked to find? Often a sketch or diagram will help you visualize the facts of the problem.
2. Pick a variable to represent the quantity to be found. Express all other quantities mentioned as expressions involving this single variable.
3. Find a way to express the quantity to be found in two different ways.
4. Write an equation showing that the two quantities found in Step 3 are equal.
5. Solve the equation.
6. Check the result in the words of the problem.

Furniture Sale

- A sofa and matching chair are on sales for $777. If the list price was $925, find the percent of markdown.

Stock Portfolio

- A college foundation owns stock in IBC (selling at $54 per share), GS (selling at $65 per share), ATB (selling at $105 per share). The foundation owns equal shares of GS and IBC, but five times as many shares of ATB.

  In this portfolio is worth $450,800, how many shares of each type does the foundation own?

Break Point Analysis

- One machine has a setup cost of \$400 and a unit cost of \$1.50, and another machine has a setup cost of \$500 and a unit cost of \$1.25. Find the break point.

Financial Planning

- A professor has \$15,000 to invest for 1 year. He invest some at 8% and then rest at 7%. If his total profit from these investments is \$1110, how much did he invest at each rate?

Absolution Value

- The **Absolution Value** of a number a, denote by the symbol $|a|$, is defined by the rules:
  - $|a| = a$ if $a \geq 0$
  - $|a| = -a$ if $a < 0$

Solving Absolute Inequality

- If k is a non-negative constant, then for any real number x:
  - $|x| > k$ is equivalent to $x < -k$ or $x > k$
  - $|x| \geq k$ is equivalent to $x \leq -k$ or $x \geq k$

Absolute Value Equation

- Solve the equation |3x – 2| = 5.

Equations with Two Absolute Values

- Solve the equation |5x + 3| = |3x + 25|.

Inequalities

- An **Inequality** is a statement in which two expressions are related by an inequality symbol.
- Statements of the form a < b or b > a are called **Strict Inequalities**.
- Statements of the form a £ b or b ³ a are called **Nonstrict Inequalities**.

Inequality Notation

Symbol	**Meaning**
<	Less than
>	Greater than
£	Less than or equal to
³	Greater than or equal to

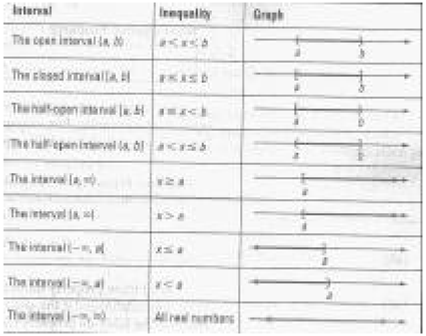
The Trichotomy Property

- If a and b are real numbers, then one of the following statement is true:
  - a < b
  - a = b
  - a > b

Intervals

- A **closed interval**, denoted by **[a, b]**, consist of all real numbers x for which a ≤ x ≤ b.
- An **open interval**, denoted by **(a, b)**, consist of all real numbers x for which a < x < b.
- The **half-open intervals** or **half-closed intervals** are **(a, b]**, consisting of all real numbers x for which a < x ≤ b and **[a, b)**, consisting of all real numbers x for which a ≤ x < b.

Graphical Representation



Interval	Inequality	Graph
The open interval (a, b)	$a < x < b$	a b
The closed interval [a, b]	$a \le x \le b$	a b
The half-open interval [a, b)	$a \le x < b$	a b
The half-open interval (a, b]	$a < x \le b$	a b
The interval [a, ∞)	$x \ge a$	a
The interval (a, ∞)	$x > a$	a
The interval (−∞, a]	$x \le a$	a
The interval (−∞, a)	$x < a$	a
The interval (−∞, ∞)	All real numbers	

Example

- Write each inequality using interval notation.
  - 1 ≤ x ≤ 3
  - -4 < x < 0
  - x > 5
  - x ≤ 1

Nonnegative Property

- For any real number a,
  - $a^2 \geq 0$

Transitive Property of Inequality

- For any real numbers a, b and c,
  - If $a < b$ and $b < c$, then $a < c$.
  - If $a > b$ and $b > c$, then $a > c$.
  - If $a \leq b$ and $b \leq c$, then $a \leq c$.
  - If $a \geq b$ and $b \geq c$, then $a \geq c$.

Addition Property of Inequality

- For any real numbers a, b and c,
  - If $a < b$, then $a + c < b + c$.
  - If $a > b$, then $a + c > b + c$.
  - If $a \leq b$, then $a + c \leq b + c$.
  - If $a \geq b$, then $a + c \geq b + c$.

Multiplication Property of Inequality

- For any real numbers a, b and c,
  - If $a < b$ and $c > 0$, then $ac < bc$.
  - If $a < b$ and $c < 0$, then $ac > bc$.
- For any real numbers a, b and c,
  - If $a > b$ and $c > 0$, then $ac > bc$.
  - If $a > b$ and $c < 0$, then $ac < bc$.

Reciprocal Property for Inequality

- For any real number a,
  - If $a > 0$, then $1/a > 0$
  - If $a < 0$, then $1/a < 0$

Solving an Inequalities

- Solve the inequality: $4x + 7 \geq 2x - 3$. Graph the solution set.

Solving Combined Inequality

- Solve the inequality: $-5 < 3x - 2 < 1$. Graph the solution set.

Using the Reciprocal Property to Solve an Inequality

- Solve the inequality: $(4x - 1)^{-1} > 0$. Graph the solution set.

Create Equivalent Inequalities

- If $-1 < x < 4$, find *a* and *b* so that $a < 2x + 1 < b$.

Inequalities with Absolute Values

- Solve the inequality $|3x + 2| \le 5$.

Quadratic Inequality

- A Quadratic Inequality is an inequality that can be written in the form
  - $ax^2 + bx + c < 0$; or
  - $ax^2 + bx + c > 0$; or
  - $ax^2 + bx + c \le 0$; or
  - $ax^2 + bx + c \ge 0$
- for any real number $a \ne 0$, b and c.

Solving Quadratic Inequality

- Solve the inequality $x^2 - x - 12 < 0$.

- Solve the inequality $2x^2 + 5x - 12 \ge 0$.