Ma104: Mathematics for Business Major

Assignment 5 (Answer)

1. **Find two numbers that have the maximum products, subject to the condition their sum is 1. [10 marks]**

Because their sum is 1, we can let x represent one number and $1 - x$ represent the other. If y represents their product, we can write the equation

$$y = x(1 - x)$$

or

$$y = -x^2 + x.$$

This equation is the equation of a parabola that opens downward. Its highest point, which is the maximum value of y, occurs when

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-1}{2(-1)} \\ &= \frac{1}{2}. \end{aligned}$$

If $x = \frac{1}{2}$, then $1 - x = \frac{1}{2}$. The two numbers are $\frac{1}{2}$ and $\frac{1}{2}$. Among the numbers with a sum of 1, they give the maximum product. So no two numbers with a sum of 1 can have a product larger than $\frac{1}{4}$. ■

2. **Solve $2^{x-1} = 3^x$ and give your answer to 1 decimal place. [10 marks]**

We first take the base-10 logarithm of each side:

$2^{x-1} = 3^x$	Original equation
$\log(2^{x-1}) = \log(3^x)$	Take log of each side.
$(x - 1)\log(2) = x \cdot \log(3)$	Power rule
$x \cdot \log(2) - \log(2) = x \cdot \log(3)$	Distributive property
$x \cdot \log(2) - x \cdot \log(3) = \log(2)$	Get all x-terms on one side.
$x[\log(2) - \log(3)] = \log(2)$	Factor out x.
$x = \dfrac{\log(2)}{\log(2) - \log(3)}$	Exact solution
$x \approx -1.7095$	Approximate solution

You can use a calculator to check -1.7095 in the original equation. As the first step of the solution, we could have taken the logarithm of each side using any base. We chose base 10 so that we could use a calculator to find an approximate solution from the exact solution. ■

3. Solve for x: log(x) + log(x – 1) = log(8x – 12) – log(2). [10 marks]

Apply the product rule to the left-hand side and the quotient rule to the right-hand side to get a single logarithm on each side:

$$\log(x) + \log(x - 1) = \log(8x - 12) - \log(2).$$

$$\log[x(x - 1)] = \log\left(\frac{8x - 12}{2}\right) \quad \text{Product rule; quotient rule}$$

$$\log(x^2 - x) = \log(4x - 6) \quad \text{Simplify.}$$

$$x^2 - x = 4x - 6 \quad \text{One-to-one property of logarithms}$$

$$x^2 - 5x + 6 = 0$$

$$(x - 2)(x - 3) = 0$$

$$x - 2 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = 2 \quad \text{or} \quad x = 3$$

Neither $x = 2$ nor $x = 3$ produces undefined terms in the original equation. Use a calculator to check that they both satisfy the original equation. The solution set is $\{2, 3\}$. ■

4. Solve the logarithmic equation: log x + log (x – 3) = 1. [10 marks]

Since $\frac{1}{2} = 2^{-1}$, we can write the equation in the form

$$2^{x^2+2x} = 2^{-1}$$

Since equal quantities with equal bases have equal exponents, we have

$$x^2 + 2x = -1$$

$$x^2 + 2x + 1 = 0 \quad \text{Add 1 to both sides.}$$

$$(x + 1)(x + 1) = 0 \quad \text{Factor the trinomial.}$$

$$x + 1 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{Set each factor equal to 0.}$$

$$x = -1 \quad | \quad x = -1$$

Verify that -1 satisfies the equation.

5. Solve the exponential equation: $2^{x^2+2x} = \frac{1}{2}$. [10 marks]

$$\log x + \log (x - 3) = 1$$

$$\log x(x - 3) = 1 \quad \text{Use Property 5 of logarithms.}$$

$$x(x - 3) = 10^1 \quad \text{Use the definition of logarithms to change the equation to exponential form.}$$

$$x^2 - 3x - 10 = 0 \quad \text{Remove parentheses and subtract 10 from both sides.}$$

$$(x + 2)(x - 5) = 0 \quad \text{Factor the trinomial.}$$

$$x + 2 = 0 \quad \text{or} \quad x - 5 = 0 \quad \text{Set each factor equal to 0.}$$

$$x = -2 \quad | \quad x = 5$$

Check: The number -2 is not a solution, because it does not satisfy the equation (a negative number does not have a logarithm). We will check the remaining number, 5.

$$\log \boldsymbol{x} + \log (\boldsymbol{x} - 3) = 1$$

$$\log \mathbf{5} + \log (\mathbf{5} - 3) \stackrel{?}{=} 1 \qquad \text{Substitute 5 for } x.$$

$$\log 5 + \log 2 \stackrel{?}{=} 1$$

$$\log 10 \stackrel{?}{=} 1 \qquad \text{Use Property 5 of logarithms.}$$

$$1 = 1 \qquad \log 10 = 1.$$

Since 5 satisfies the equation, it is a solution. ■