Matrices

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Matrix

- A **Matrix** is a rectangular table of numbers.
- The numbers of a matrix are called **Entries** (or **Elements**) of the matrix.
- The Dimension of the matrix is the number of rows and columns its contains. The dimension is written in the form:

  (Number of rows) × (Number of columns)

  where the × is read "by".

Example

- Find a_{23}, a_{31} and a_{12} for the following matrix:

$$A = \begin{bmatrix} 6 & -1 & 3 \\ 4 & 2 & 5 \\ 7 & 1 & 8 \end{bmatrix}$$

- Answer
  - $a_{23} = 5$, $a_{31} = 7$ and $a_{12} = -1$.
- (Do Ex. 1)

Algebraic Operations

Algebraic Operations on Matrices

Equality	Two matrices A and B are equal if they have the same dimension and corresponding entries are equal, that is $a_{ij} = b_{ij}$ for all pairs (i, j)
Addition	Two matrices may be added only if they have the same dimension. To add two matrices, add corresponding entries.
Subtraction	Two matrices may be subtracted only if they have the same dimension. To subtract two matrices, subtract corresponding entries.
Scalar Multiplication	To multiply a matrix by a scalar, multiply each entry of the matrix by the scalar.

Basic Properties of Matrix Addition and Scalar Multiplication

Basic Properties of Matrix Addition and Scalar Multiplication

If A, B, and C have the same dimensions, the following laws hold:

1. **Associative property of addition:** $(A+B)+C=A+(B+C)$
2. **Commutative property of addition:** $A+B=B+A$
3. **Distributive properties of scalar multiplication:**
   a. $k(A+B)=kA+kB$
   b. $(k+h)A=kA+hA$
4. **Properties of the zero matrix** (the matrix having all entries equal to zero):
   a. $A_{m\times n}+0_{m\times n}=A_{m\times n}$
   b. $k\cdot 0_{m\times n}=0_{m\times n}$
   c. $0_{p\times m}\cdot A_{m\times n}=0_{p\times n}$

Example

If $A=\begin{bmatrix} 2 & {}^-1 & 4 \\ {}^-3 & 7 & 10 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 5 & {}^-1 \\ 12 & 2 & 6 \end{bmatrix}$, and $C=\begin{bmatrix} 1 & 1 & {}^-4 \\ 2 & 8 & 17 \end{bmatrix}$

Calculate each of the following:

a. $A+B$ b. $A-C$ c. $3B$ d. $A+2B-C$ e. $B+5C$

Answer

a. $A+B=\begin{bmatrix} 2 & {}^-1 & 4 \\ {}^-3 & 7 & 10 \end{bmatrix}+\begin{bmatrix} 3 & 5 & {}^-1 \\ 12 & 2 & 6 \end{bmatrix}=\begin{bmatrix} 2+3 & {}^-1+5 & 4+({}^-1) \\ {}^-3+12 & 7+2 & 10+6 \end{bmatrix}=\begin{bmatrix} 5 & 4 & 3 \\ 9 & 9 & 16 \end{bmatrix}$

b. $A-C=\begin{bmatrix} 2 & {}^-1 & 4 \\ {}^-3 & 7 & 10 \end{bmatrix}-\begin{bmatrix} 1 & 1 & {}^-4 \\ 2 & 8 & 17 \end{bmatrix}=\begin{bmatrix} 2-1 & {}^-1-1 & 4-({}^-4) \\ {}^-3-2 & 7-8 & 10-17 \end{bmatrix}=\begin{bmatrix} 1 & {}^-2 & 8 \\ {}^-5 & {}^-1 & {}^-7 \end{bmatrix}$

c. $3B=3\begin{bmatrix} 3 & 5 & {}^-1 \\ 12 & 2 & 6 \end{bmatrix}=\begin{bmatrix} 3\cdot 3 & 3\cdot 5 & 3({}^-1) \\ 3\cdot 12 & 3\cdot 2 & 3\cdot 6 \end{bmatrix}=\begin{bmatrix} 9 & 15 & {}^-3 \\ 36 & 6 & 18 \end{bmatrix}$

d. $A+2B-C=\begin{bmatrix} 2 & {}^-1 & 4 \\ {}^-3 & 7 & 10 \end{bmatrix}+2\begin{bmatrix} 3 & 5 & {}^-1 \\ 12 & 2 & 6 \end{bmatrix}-\begin{bmatrix} 1 & 1 & {}^-4 \\ 2 & 8 & 17 \end{bmatrix}$

$=\begin{bmatrix} 2+6-1 & {}^-1+10-1 & 4-2+4 \\ {}^-3+24-2 & 7+4-8 & 10+12-17 \end{bmatrix}=\begin{bmatrix} 7 & 8 & 6 \\ 19 & 3 & 5 \end{bmatrix}$

e. $B+5C=\begin{bmatrix} 3 & 5 & {}^-1 \\ 12 & 2 & 6 \end{bmatrix}+5\begin{bmatrix} 1 & 1 & {}^-4 \\ 2 & 8 & 17 \end{bmatrix}=\begin{bmatrix} 3+5 & 5+5 & {}^-1-20 \\ 12+10 & 2+40 & 6+85 \end{bmatrix}=\begin{bmatrix} 8 & 10 & {}^-21 \\ 22 & 42 & 91 \end{bmatrix}$

Vector

- A **Vector** is a list of numbers.
- If the list is written horizontally in a row, then it is called a **Row Vector**.
- If the list is written vertically in a column, then it is called a **Column Vector**.
- A column vector is not equal to a row vector even if there entries are the same.

Example of Row and Column Vector

An example of a row vector is $r = [15\ \ {}^{-}3\ \ 4]$

An example of a column vector is $r = \begin{bmatrix} 15 \\ {}^{-}3 \\ 4 \end{bmatrix}$

Type of Matrices

- A Matrix is a **Square Matrix** if the number of rows equal to to the number of column.
- The **Diagonal** of a matrix A consists of all the entries a_{ii}.
- A **Diagonal Matrix** is a square matrix whose only diagonal entries are nonzero.
- The **Identity Matrix** of order n is the n×n matrix that the diagonal entries are 1 and others are 0. The identity matrix is denoted by I_n.
- The **Zero Matrix** is an m×n matrix that all entries are zero. The zero matrix is denoted by 0_{mn}.

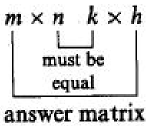
Example

The matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is the identity matrix $\mathbf{I}$

The matrix $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the zero matrix $\mathbf{0}_{34}$

Matrix Multiplication

- In order to multiple AB, the number of columns of A must be equal to the number of rows of B. The answer matrix will have the same number of rows as A and the same number of columns as B.
- To find the (i, j) entry of the answer matrix, multiple row i of A into column of B and add.
- If the middle numbers of AB are equal, the dimension of AB is given by the two outside numbers. Thus, if n = k, the answer is an m × h matrix.



Example

What is $[3\ \ 7\ \ 2]\begin{bmatrix} 8 \\ ^{-}1 \\ 6 \end{bmatrix}$?

$$[3\ \ 7\ \ 2]\begin{bmatrix} 8 \\ ^{-}1 \\ 6 \end{bmatrix} = 3\cdot 8 + 7\cdot(^{-}1) + 2\cdot 6 = 24 - 7 + 12 = 29.\ \text{Thus}\ [3\ \ 7\ \ 2]\begin{bmatrix} 8 \\ ^{-}1 \\ 6 \end{bmatrix} = [29]$$

- (Do Ex. 2)

Transforming Matrices

- If A is an m×n matrix, then the **Transpose** of A is the n×m matrix whose first row is the first column of A, whose second row is the second column of A and so on.
- The transpose of A is denoted by A^T.

Example

Find the transpose of $A = \begin{bmatrix} 3 & 1 & ^{-}5 \\ 12 & 7 & 6 \end{bmatrix}$

Since A is a 2×3 matrix, the dimension of A^t is 3×2. The first row of A^t is the first column of A, that is, 3 12. The second row of A^t is the second column of A, that is, 1 7. The third row of A^t is the third column of A. Thus, we have

$$A^t = \begin{bmatrix} 3 & 12 \\ 1 & 7 \\ ^{-}5 & 6 \end{bmatrix}$$

Properties of the Transpose

- The transpose satisfies the following properties:
  - $(A^T)^T = A$
  - $(A + B)^T = A^T + B^T$
  - $(AB)^T = B^T A^T$
  - $(kA)^T = kA^T$
- Matrix A is **Symmetric** if $A = A^T$.