# Graph Theory

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# Graph

- A **Graph** (or undirected graph) G consists of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge $e \in E$ is associated with an unordered pair of vertices.
- In simple, a graph is a set of Points (called **Vertices**) connected by Lines (called **Edges**).
- Graphs are denoted by uppercase letters such as G. Then the set of vertices of a graph G is denoted by **V(G)**, and the set of edges of a graph G is denoted by **E(G)**.



# Adjacent, Nonadjacent, Incident

- If there is an edge joining a pair of vertices, those vertices are said to be **Adjacent**. Otherwise, they are **Nonadjacent**.
- An edge is **Incident** with a vertex if the edge is joined to the vertex.



# Degree of Graph

- The number of edges connected to a given vertex is called the **Degree** of that vertex and denoted by d(v).
- The **Degree Sequence** of a graph is the list of the degree of its vertices in non-increasing order.
- (Do Ex. 1 – 2)

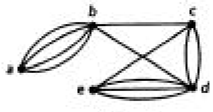
# Sum of Degree

- In a graph G, the sum of the degree of the vertices equals twice the number of edges.
- The sum of the degree if the vertices of a graph is an even number.

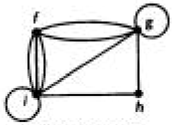


# Multigraphs and Pseudographs

- A **Multigraph** allow more than one edge between a pair of vertices. Such edges are called **Multiple Edges**.
- A **Pseudographs** allow loops and multiple edges. A **Loop** is an edge that connects a vertex to itself.



A multigraph

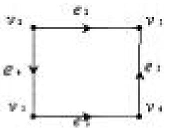


A pseudograph



# Directed Graph

- A **Directed Graph** (or Digraph) G consists of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge $e \in E$ is associated with an ordered pair of vertices.





# Parallel and Cycle

- The directed edges are called **Arcs**.
- **Parallel Arcs** are pair of arcs in which one is directed from vertex a to vertex b, and the other in directed from vertex b to vertex a.
- A **Cycle** is path that begins and ends at the same place. A cycle having n vertices is denoted $C_n$.
- (Do Ex. 3 – 4)

# Indegree and OutDegree

- The Indegree of a vertex v is the number of arcs directed toward v and is denoted by id(v).
- The Outdegree of a vertex v is the number of arcs directed away from v and is denoted by od(v).
- In a directed graph, the sum of indegree is equal to the sum of outdegree.
- (Do Ex. 5 – 6)



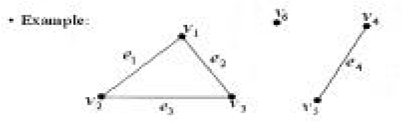
# Path

- A Path in a graph G is a sequence of distinct vertices $v_0, v_1, v_3, \ldots v_k$, such that $v_0v_1, v_1v_2, \ldots v_{k-1}v_k$ are edges of G.
- The Length of a path of a graph is the number of edges in it.
- A path having n vertices is denoted by $P_n$.
- A path using k distinct vertices has length k-1.



# Connected & Disconnected Graph

- A graph is **Connected** if every pair of its vertices is connected by a path.
- A graph is **Disconnected** if there is not a path between every pair of vertices.
- A graph that is disconnected contains two or more pieces, called Components of the graph.



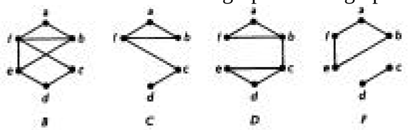



# Subgraphs

- A Subgraph A of graph B has as its vertex set a subset of the vertices of B, and as the edge set a subset of the edges of B.
- An **Included Subgraph** is whenever an edge appears between a pair of vertices in the original graph it also must appear in the subgraph.

# Example

- Determine which of the graph are subgraphs of graph B.




Graph $B$ is a subgraph of graph $B$ since $B \subseteq B$

Graph $C$ is a subgraph of graph $B$: A subgraph need not contain all the vertices of the original graph.

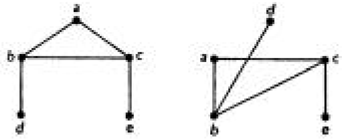
Graph $D$ is not a subgraph of graph $B$ since $bc \in E(D)$, but $bc \notin E(B)$

Graph $F$ is a subgraph of graph $B$: A subgraph need not contain all the edges of the original graph.



# Isomorphic Graph

- Graphs G and H are **Isomorphic** if there is a 1-1, onto function f from the vertices of G to the vertices of H and 1-1, onto function g from the edges of G to the edges of H, so that an edges e is incident on u and v in G if and only if the edge g(e) is incident on f(u) and f(w) in H.
- The pair of functions f and g is called **Isomorphism** of G onto H.





# Isomorphic Graph

- If two graphs are isomorphic, corresponding vertices have the same degree.
- Let G and H be isomorphic with isomorphism $f:V(G) \rightarrow V(H)$. If $v_1, v_2, \ldots v_k$ is a shortest path between vertices $v_1$ and $v_k$ in G, then $f(v_1), f(v_2), \ldots f(v_k)$ is a shortest path between vertices $f(v_1)$ and $f(v_k)$ in H.

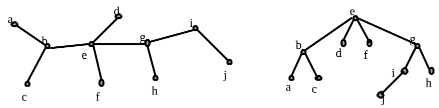


# Isomorphic Graph Checking

- Number of vertices
- Number of components.
- Number of edges.
- Degree sequence.
- Length of the shortest path between pairs of vertices with a given degree.
- Length of the longest path in graph.

# Trees

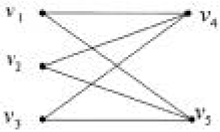
- A **Tree** is a connected graph that does not contain a cycle as a subgraph.
- A **Rooted Tree** is a tree in which a particular vertex is designated the root.






# Bipartite Graph

- A graph is **Bipartite** if its vertices can be separated in to two sets A and B, so that vertices within the same set are nonadjacent.



- (Do Ex. 7)



# Bipartite Graph Checking

1. Label any vertex a.
2. Label all vertices adjacent to a with the label b.
3. Label all vertices that are adjacent to a vertex just labeled b with label a.
4. Repeat step 2 and 3 until you have labeled all vertices with a distinct label (bipartite) or you have a conflict (not bipartite).



# Complete Graph

- A **Complete Bipartite Graph** is one in which each vertex in set A is adjacent to every vertex in set B.
- The complete bipartite graph having m vertices in A and n vertices in B is denotes $K_{m,n}$.
- The **Complete Graph** $K_n$ has n vertices, with every vertex connect to every other vertex.
- (Do Ex. 8)

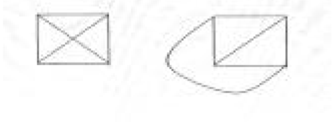
# Regular Graph

- A graph is k-regular if all its vertices have the same degree k.
- $K_n$ is (n-1) regular because each vertex of $K_n$ has degree n-1.
- $C_n$ is 2-regular.



# Planar Graph

- A graph is Planar if it can be drawn in the plane without its edges crossing.
- Example
  - Is $K_4$ a planar graph? YES





# Euler's Formula

- In a planar graph, each cycle not containing any smaller cycles enclose a region called a **Face**. The region exterior to the graph is called the **Infinite Face**.
- In a connected plan graph with f faces, e edges, and n vertices, we have the relationship
  - $n - e + f = 2$



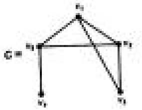
# Adjacency Matrix

- A **(0, 1)-Matrix** is a matrix each of whose entries is 0 or 1. Examples are the identity matrix and the zero matrix.
- Let the vertices of graph G be labeled $v_1, v_2, \dots v_n$. The **Adjacency Matrix A(G)** is the n×n (0, 1)-matrix:

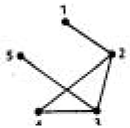
$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

- Sine a vertex is never adjacent to itself, A(G) has 0's on the diagonal.

# Example

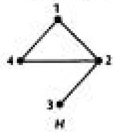


$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



$$A(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$A(H) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$






# Incidence Matrix

- A graph G has vertex set $V(G)=\{v_1, v_2, \dots v_n\}$ and edge set $E(G) =\{e_1, e_2, \dots e_m\}$. The **Incidence Matrix** M(G) is the $n\times m$ (0, 1)-matrix defined by:

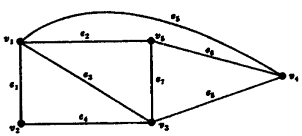
$$m_{ij} = \begin{cases} 1 & \textbf{if } v_i \textbf{ is an endpoint of } e_j \\ 0 & \textbf{otherwise} \end{cases}$$

- Since an edges has two end points, there are even number of 1 in each column of M(G).
- Each 1 in row i of M(G) corresponds to an edge incident with $v_1$. The number of 1's in row i is the degree of $v_i$.



# Example

- Find the Incidence Matrix of the graph G.



$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

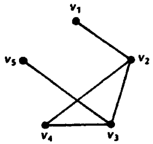


# Distance Matrix

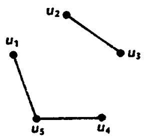
- The distance between $v_i$ and $v_j$, denoted $d_{ij}$, is the length of the shortest path connecting $v_i$ and $v_j$.
  - If graph G is connected, $d_{ij}$ is finite for every pair $v_i$, $v_j$.
  - If graph G is disconnect, $d_{ij} = \infty$ for vertices in distinct components and is finite otherwise.
- The distance matrix D(G) is the $n\times n$ matrix where $[D(G)]_{ij} = d_{ij}$.
- The distance matrix is always symmetric.

# Example

- Find the distance matrix for the graphs G and H.




G




H

$$D(G)=\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ 3 & 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

$$D(H)=\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array}\begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 0 & \infty & \infty & 2 & 1 \\ \infty & 0 & 1 & \infty & \infty \\ \infty & 1 & 0 & \infty & \infty \\ 2 & \infty & \infty & 0 & 1 \\ 1 & \infty & \infty & 1 & 0 \end{bmatrix} \end{array}$$



# Walk, Path and Trail

- A **Walk** is a sequence of vertices $v_0$, $v_1$, … $v_k$ where $v_iv_{i+1}$ is an edge of G.
- A **Trail** is a walk in which the edges are distinct.
- A **Path** requires that vertices and edges be distinct.
- Therefore, a path is a special type of trail, and a trail is a special type of walk.



# Eulerian Trail

- An **Eulerian Trail** is a trail that includes each edge of the graph.
- If the tail begins and ends at the same vertex, it said to be **Closed**.
- A graph is **Eulerian** if it contains a closed Eulerian Trail. In other word, a graph is Eulerian if it contains a walk that includes each edges exactly once and ends at the starting vertex.
- A graph is **Semi-Eulerian** if it contains an Eulerian Tail.



# Eular Theorem

- A Connected graph is Eulerian if and only if each vertex has an even degree.
- A connected graph is semi-Eulerian but not Eulerian if and only if the graph contains precisely two vertices having odd degree. Furthermore, the Eulerian trail must begin at one of the odd vertices and end at the other.

## Fleury's Algorithm

- If a graph is Erulerian. To find an Eularian trail, begin at any vertex. Record and erase each edge as it is used, subject to the following condition: Never use a bridge unless there is no alternative.



## Hamiltonian Graphs

- A path that contains every vertex of a graph is called **Hamiltonian Path**.
- If a graph contains a Hamiltonian path it is called **Semi-Hamiltonian**.
- A **Hamiltonian Cycle** is a cycle that includes every vertex. If a graph contains a Hamiltonian cycle, then the graph is **Hamiltonian**.