Probability

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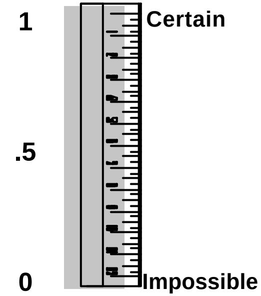
What is Probability?

- **Probability** is the numerical measure of likelihood that the event will occur.
  - Simple Event
  - Joint Event
  - Compound Event
- Lies between 0 & 1
- Sum of events is 1



Sample Spaces and Events

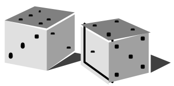
- Sample Space
  - Complete Collection of Outcomes (the set S)
- Simple Event (Sample Point)
  - Outcome With 1 Characteristic; a particular outcome, i.e, an element in S.
- Joint Event
  - 2 Events Occurring Simultaneously
- Compound Event
  - One or Another Event Occurring
- Impossible Event
  - The empty set $\varnothing$ in S.

Simple Event

A. Female
B. Under age 20, < 20
C. Has 3 credit cards
D. Red card from a deck of bridge cards
E. Ace card from a deck of bridge cards

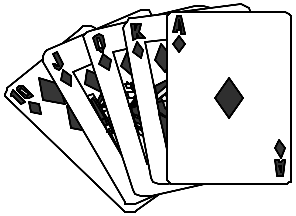
Joint Event

- **A and B**, $(A \cap B)$:
  - Female, Under age 20
- **D and E**, $(D \cap E)$:
  - Red, ace card from bridge deck



Compound Event

- **D or E**, $(D \cup E)$:
  - Ace or Red card from bridge deck



Example

- The sample space S consist of the 6 possible numbers, S = {1, 2, 3, 4, 5, 6}
  - Event that an even number occurs, A = {2, 4, 6}
  - Event that an odd number occurs, B = {1, 3, 5}
  - Event that a prime number occurs, C = {2, 3, 5}
  - Event that an even number or a prime number occurs, $A \cup C = \{2, 3, 4, 5, 6\}$
  - Event that an odd prime number occurs, $B \cap C = \{3, 5\}$
  - Event that a prime number does not occurs, C'={1, 4, 6}

Theorems of Probability

- Let S be a sample space, let E be the class of events, and let P be a real-value function defined on E. Then P is called a Probability Function, and P(A) is called the probability of the event A if the following hold:
  - For every event A, $0 \le P(A) \le 1$
  - P(S) = 1
  - If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$
  - If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events, then $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$

Theorems of Probability

- For two events A and B,
  - P (∅) = 0
  - P(A') = 1 – P(A)
  - If $A \subseteq B$, then $P(A) \le P(B)$
  - $P(A \setminus B) = P(A) - P(A \cap B)$
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Type of Sample Spaces

- Finite Probability Spaces
  - A finite probability space is one in which the set S contains a finite number of elements.
- Finite Equiprobable Spaces
  - A probability space is equiprobable if it is finite and the probability mass are all equal.

Example

- Three horses A, B, and C are in a race; A is twice as likely to win as B and B is twice as likely to win as C. What are their respective probabilities of winning, i.e. P(A), P(B) and P(C)?

Let $P(C) = p$; since B is twice as likely to win as C, $P(B) = 2p$; and since A is twice as likely to win as B, $P(A) = 2P(B) = 2(2p) = 4p$. Now the sum of the probabilities must be 1; hence $p + 2p + 4p = 1$ or $7p = 1$ or $p = \frac{1}{7}$. Accordingly, $P(A) = 4p = \frac{4}{7}$, $P(B) = 2p = \frac{2}{7}$, $P(C) = p = \frac{1}{7}$

Example

- Let a card be selected at random from an ordinary pack of 52 cards. Let A= {the card is a spade} and B = {the card is a face card}. What is P(A), P(B) and P(A∩B)?

Since we have an equiprobable space,

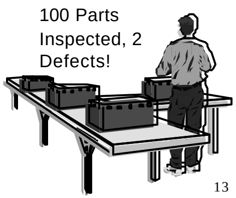
$$P(A) = \frac{\text{number of spades}}{\text{number of cards}} = \frac{13}{52} = \frac{1}{4}$$

$$P(B) = \frac{\text{number of face cards}}{\text{number of cards}} = \frac{12}{52} = \frac{3}{13}$$

$$P(A \cap B) = \frac{\text{number of spade face cards}}{\text{number of cards}} = \frac{3}{52}$$

Probability of Simple Event

- P(Event) = X / T
  - X = number of outcomes in which the event occurs
  - T = total number of possible events



Conditional Probability

- The conditional probability that B occurs, given that A occurred, is defined by

$$P(B|A) = \frac{P(AB)}{P(A)}$$

- Chain Rule for Conditional Probabilities

$$P(AB) = P(B|A)P(A)$$

Example

Let a pair of fair dice be tossed. If the sum is 6, find the probability that one of the dice is 2. In other words, if

$E = \{\text{sum is } 6\} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$ and

$A = \{\text{a 2 appears on at least one die}\}$ find $P(A|E)$ and $P(A)$

Now E consists of five elements and two of them, $(2,4)$ and $(4,2)$, belong to A: $A \cap E = \{(2,4),(4,2)\} = \frac{2}{36}$ and $P(E) = \frac{5}{36}$

Then $P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{2/36}{5/36} = \frac{2}{5}$

Multiplication Theorem

- For any events, $A_1, A_2, \ldots A_n$,
  $P(A_1 \cap A_2 \cap \ldots \cap A_n)$

$= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \ldots P(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1})$

Example

- A lot contains 12 items of which 4 are defective. Three items are drawn at random from the lot one after the other. Find the probability p that all three are not defective.

The probability that the first item is not defective is $\frac{8}{12}$ since 8 of 12 items are not defective. If the first item is not defective, then the probability that the next item is not defective is $\frac{7}{11}$ since only 7 of the remaining 11 items are not defective. If the first two items are not defective, then the probability that the last item is not defective is $\frac{6}{10}$ since only 6 of the remaining 10 items are now not defective. Thus by the multiplication theorem, $p = \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} = \frac{14}{55}$

Joint Probability Using Contingency Table

	Event		
Event	**B_1**	**B_2**	**Total**
A_1	$P(A_1 \cap B_1)$	$P(A_1 \cap B_2)$	$P(A_1)$
A_2	$P(A_2 \cap B_1)$	$P(A_2 \cap B_2)$	$P(A_2)$
Total	$P(B_1)$	$P(B_2)$	1

Joint Probability

Marginal (Simple) Probability

Example

- Find the probability for a joint event that a female, and age under 20 is drawn.

	<20	>20	**Total**
Female	47	16	63
Male	45	22	67
Total	92	38	130

Simple Event

Count Total

S = {F,<20; F,>20; M,<20; M,>20}

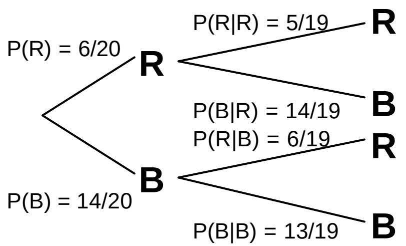
Sample Space

Stochastic Process & Tree Diagrams

- A (Finite) sequence of experiments in which each experiment has a finite number of outcomes was given probabilities is called a (finite) **Stochastic Process**.
- A convenient way of describing such a process and computing the probability of any event is by a **Tree Diagram**.

Example

- Find the probability of selecting 2 pens from 20 pens which 14 in blue and 6 in red and not replace.



Independence

- An event B is said to be independent of an event A if the probability that B occurs is not influenced by whether A has or has not occurred.
  - Event A and B are independent if
    - $P(A \cap B) = P(A)P(B)$;
  - Otherwise they are dependent.

Example

- Let a fair coin be tossed three times; we obtain the equiprobable space S= {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}. Consider the events A = {first toss is heads}, B = {second toss is heads}, C = {exactly two heads are tossed in a row}. Show that A and B and A and C are independent events, and that B and C are dependent events.

Answer

We have $P(A) = P(\{HHH, HHT, HTH, HTT\}) = \frac{4}{8} = \frac{1}{2}$

$P(B) = P(\{HHH, HHT, THH, THT\}) = \frac{4}{8} = \frac{1}{2}$

$P(C) = P(\{HHT, THH\}) = \frac{2}{8} = \frac{1}{4}$

Then $P(A \cap B) = P(\{HHH, HHT\}) = \frac{1}{4}$

$P(A \cap C) = P(\{HHT\}) = \frac{1}{8}$

$P(B \cap C) = P(\{HHT, THH\}) = \frac{1}{4}$

Accordingly, $P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A \cap B)$, and so A and B are independent;

$P(A)P(C) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} = P(A \cap C)$, and so A and C are independent;

$P(B)P(C) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} \neq P(B \cap C)$, and so B and C are dependent.

Repeated Trials

- Let S be a finite probability space. By n-Independent or Repeated Trials, we mean the probability space T consisting of ordered n-tuples of elements of S with the probability of an n-tuple defined to be the product of the probabilities of its components:
  - $P(S_1, S_2, \ldots, S_n) = P(S_1)P(S_2) \ldots P(S_n)$

Example

- Whenever three horses a, b, and c race together, their respective probabilities of winning are 1/2, 1/3 and 1/6. In other words, S = {a, b, c} with P(a) = 1/2, P(b) = 1/3 and P(c) = 1/6. If the horses race twice, what is the probability of horse a winning the first race and horse c winning the second race?

The sample space is $T = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$ for the 2 repeated trials, where to simplify things we write ac for the ordered pair (a, c). The probability of ac is $P(ac) = P(a)P(c) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$

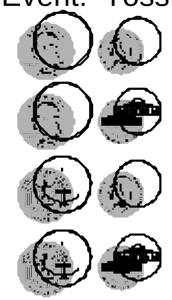
Random Variables

- A Random Variable X is a rule that assigns a numerical value to each outcome in a sample space S.

Experiment	Random Variables	Possible Values
Make 100 Sales Calls	No. of Sales	0, 1, 2, ... , 100
Inspect 70 Radios	No. of Defective	0, 1, 2, ... , 70
Answer 33 Questions	No. of Correct	0, 1, 2, ... , 33
Count Cars at Toll between 11:00 & 1:00	No. of Car Arriving	0, 1, 2, ... , n

Distribution

Event: Toss 2 Coins. Count No. of Tails.



Probability Distribution

Values, X_i	Probabilities, $P(X_i)$
0	1/4 = .25
1	2/4 = .50
2	1/4 = .25

Expectation

- Expected Value
  - Mean of Probability Distribution
  - Weighted Average of All Possible Values
  - $\mu = E(X) = \Sigma X_i P(X_i)$
- Variance
  - Weighted Average Squared Deviation about Mean
  - $\sigma^2 = E[\ (X_i - \mu)^2\ ] = \Sigma\ (X_i - \mu)^2\ P(X_i)$

Example

- In a game a player is paid $5 if he gets all heads or all tails when three coins are tossed, and he will pay out $3 if either one or two heads show. What is his expected gain (or loss) per game?

Answer

- Three coins are tossed, the sample space
  - S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}.
- Assume unbiased coins, so any one of these eight outcomes have equal chance of, say,
  - P(HHT) = P(H)P(H)P(T) = (1/2)*(1/2)*(1/2) = 1/8.
- Let E1 = {HHH, TTT}, and E2 = {the other 6 outcomes}.
  - P{E1} = 1/8 + 1/8 = 1/4, and P{E2} = 1- 1/4 = 3/4.
- It follows that the expected value is
  - μ = $5*(1/4) - $3*(3/4) = -$1.
- So the player is expected to lose one dollar per game.

Example

A coin weighted so that $P(H) = \frac{2}{3}$ and $P(T) = \frac{1}{3}$ is tossed three times. The probabilities of the points in the sample space $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ are as follows:

$P(HHH) = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{8}{27}$ $\quad P(THH) = \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{4}{27}$ $\quad P(HHT) = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} = \frac{4}{27}$

$P(THT) = \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} = \frac{2}{27}$ $\quad P(HTH) = \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{2}{3} = \frac{4}{27}$ $\quad P(TTH) = \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{2}{3} = \frac{2}{27}$

$P(HTT) = \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{2}{27}$ $\quad P(TTT) = \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{1}{27}$

Let X be the random variable which assigns to each point in S the largest number of successive heads which occurs. Find the distribution f of X.

Answer

The distribution f of X is:

$$f(0) = P(\text{TTT}) = \frac{1}{27}$$
$$f(1) = P(\{\text{HTH}, \text{HTT}, \text{THT}, \text{TTH}\}) = \frac{4}{27} + \frac{2}{27} + \frac{2}{27} + \frac{2}{27} = \frac{10}{27}$$
$$f(2) = P(\{\text{HHT}, \text{THH}\}) = \frac{4}{27} + \frac{4}{27} = \frac{8}{27}$$
$$f(3) = P(\text{HHH}) = \frac{8}{27}$$

Which is put in a table as:

x_i	0	1	2	3
$f(x_i)$	$\frac{1}{27}$	$\frac{10}{27}$	$\frac{8}{27}$	$\frac{8}{27}$

Example

- A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective ones. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

Answer

Let X = no. of good components in the sample. The Probability distribution of X is given by

$$f(x) = \frac{C_x^4 C_{3-x}^3}{C_3^7}$$

Where x = 0,1,2,3

So f(0) = 1/35, f(1) = 12/35, f(2) = 18/35, f(3) = 4/35 (Note the sum of $f(x_i)$ = 1.0)

Then μ = E(X) = (0) (1/35) + (1) (12/35) + (2) (18/35) + (3) (4/35) = 1.7

This means that if a sample size of 3 is selected at random over and over again from a lot of 4 good components and 3 defective components, it would contain, on the average, 1.7 good components.

Binomial Distribution Properties

- Two Different Sampling Methods
  - Infinite Population Without Replacement
  - Finite Population With Replacement
- Sequence of n Identical Trials
- Each Trial has 2 Outcomes
  - 'Success' (Desired Outcome) or 'Failure'
- Constant Trial Probability
- Trials are Independent

Binomial Probability Distribution Function

Pick x from n $\binom{n}{k}$ — Probability of x successes

$$P(X = k \mid n,p) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

Probability of (n-k) failures

$P(X = k \mid n,p)$ = probability that $X = k$ given n & p

n = *sample* size

p = probability of 'success'

k = number of 'successes' in *sample* (X = 0, 1, 2, ..., n)

Example

- A fair die is tossed 180 times. Find the expected number of sixes, and the standard deviation.

The expected number of sixes is $\mu = np = 180 \cdot \frac{1}{6} = 30$. The standard deviation is $\sigma = \sqrt{npq} = \sqrt{180 \cdot \frac{1}{6} \cdot \frac{5}{6}} = 5$

Example

- Toss 1 Coin 4 times in a row. What's the Probability of 3 tails?

$$P(X = x \mid n,p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$P(X = 3 \mid 4,.5) = \frac{4!}{3!(4-3)!} .5^3 (1-.5)^{4-3}$$

$$= .25$$

Bayes' Theorem Formula

$$P(B_i \mid A) = \frac{P(A \mid B_i) \cdot P(B_i)}{P(A \mid B_1) \cdot P(B_1) + \cdots + P(A \mid B_k) \cdot P(B_k)}$$

$$= \frac{P(B_i \cap A)}{P(A)}$$

Same Event

All B_i's are the same event (e.g. B_2)!

Example

- A box contains 12 items of which 4 are in red color. Three items are drawn at random from the box one after the other. Find the probability that all three are not in red color.
- Probability = (8/12)(7/11)(6/10) = 14/55