Combinatorics

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Introduction

- **Combinatorial Analysis**, or **Combinatorics**, which includes the study of permutations, combinations, and partitions, is concerned with determining the number of logical possibilities of some event without necessarily enumerating each case.

The Sum Rule

- If something can happen in n_1 ways, and a 2nd things can happen in n_2 ways, … and a 3rd things can happen in n_3 way, …. Then there are $n_1 + n_2 + n_3 + \ldots$ ways in exactly one of the events can occur.
- Example:
  - If there are 3 different courses offered in the morning and 2 different course offered in the afternoon.
    - There will be 3 + 2 choices for a student to enroll in only one course.

The Product Rule

- If something can happen in n_1 ways, and a 2nd things can happen in n_2 ways, … and a 3rd things can happen in n_3 way, …. Then all the things together can happen in $n_1 \times n_2 \times n_3 \times \ldots$ ways.
- Example:
  - If there are 3 different courses offered in the morning and 2 different course offered in the afternoon.
    - There will be 3×2 choices for a student to enroll 1 in morning and 1 in afternoon.

Factorial Notation

- n! denoted the product of the positive integer from 1 to n, inclusive: $n! = n(n-1)(n-2)\ldots 3\times 2\times 1$
  - $0! = 1$
  - $1! = 1$
  - $n! = n(n-1)!$

Counting Formulas

	Order Matters	**Order not Matter**
Elements Repeated	K-sample	K-selection
Elements not Repeated	K-permutation	K-combination

K-samples (Repeat + Order)

- With a k-sample order of the elements matters and elements can be repeated. The formula for a k-sample is $\mathbf{n^k}$, where k is the number of samples you select from the set of n elements.
- Example:
  - Find the number of combination that 2 letters can be drawn from the letter (A, B, C, D) with replacement.
  - Answer: $4^2 = 16$ ways
    - AA AB AC AD BA BB BC BD
    - CA CB CC CD DA DB DC DD

K-permutations (No-repeat + Order)

- With a k-permutation the order of the elements matters, but repetition of the elements is not allowed.
- The formula for a k-permutation is $$P(n,k) = \frac{n!}{(n-k)!}$$
- Example:
  - Find the number of combination that 2 letters can be draw from 4 letters (A, B, C, D) without replacement.
  - Answer: P(4, 2) = 12
    - AB AC AD BA BC BD
    - CA CB CD DA DB DC

K-combinations (No-repeat + No-order)

- With a k-combination the order in which elements are selected does not matter and the elements cannot be repeat.
- The formula for a k-combination is $C(n,k) = \frac{n!}{(n-k)!k!}$
- Example:
  - Find the number of combination that 2 distinct letters can be drawn from 4 letters (A, B, C, D) one by one without replacement.
  - Answer: C(4, 2) = 6
    - AB AC AD BC BD CD

K-selections (Repeat + No-order)

- With a k-selection the order in which elements are selected does not matter and the elements can be repeat.
- The formula for a k-selections is C_k^{n+k-1}
- Example:
  - Find the number of combination that 2 letters can be drawn from 4 letters (A, B, C, D) one by one with replacement.
  - Answer: C(4+2-1, 2) = 10
    - AA AB AC AD BB
    - BC BD CC CD DD

Example

- A relation maps elements from set A to set B. A has m elements, and B has n elements. How many *one-to-one functions* are there if:
  - $m = n$?
  - $m > n$?
  - $m < n$?

Answer

- $m = n$
  - Each element of A must be mapped to exactly one element of B; no element of B may be mapped to from more than one element of A. Thus, there are n different possibilities for the image of the first element of A, $n-1$ different possibilities for the image of the second element of A, and so on. Therefore the answer is $n(n-1)(n-2)\ldots1 = n!$.
- $m > n$
  - There are insufficient elements for us to construct any one-to-one functions. Therefore the answer is 0.
- $m < n$
  - The answer here is a generalization of the case when $m = n$, since once we have picked the image of the final element of A, we still have $n - m$ elements left over. There the answer is $n(n-1)(n-2)\ldots(n-m+1) = nPm$.

Pigeonhole Principle

- Pigeonhole Principle
  - If n pigeonholes are occupied by n+1 or more pigeons, then at least 1 pigeonhole is occupied by more than 1 pigeon.
- Generalized Pigeonhole Principle
  - If n pigeonholes are occupied by kn+1 or more pigeons, where k is positive integer, then at least 1 pigeonhole is occupied by more than k+1 pigeon.

Example:

- Find the minimum number of students in a class to be sure that three of them are born in the same month.
- Here n = 12 are the pigeonholes and k+1=3. Hence among any kn+1= 25 students (pigeons), three of them are born in the same month.

The Inclusion-Exclusion Principle

- For any finite set A, B, C
  - $|A \cup B| = |A| + |B| - |A \cap B|$
  - $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| - |A \cap B \cap C|$

Example

- Find the number of mathematics students at a university taking at least one of the language Mandarin, English and Japan given the following data:
  - 65 study Mandarin
  - 45 study English
  - 42 study Japanese
  - 20 study Mandarin and English
  - 25 study Mandarin and Japanese
  - 8 study all three languages

Answer

- We want to find $|M \cup E \cup J|$ where M, E and J denote the sets of students studying Mandarin, English and Japanese respectively.
- By the inclusion-exclusion principle,

$|M \cup E \cup J|$

$= |M| + |E| + |J| - |M \cap E| - |M \cap J| - |E \cap J| - |M \cap E \cap J|$

$= 65 + 45 + 42 - 20 - 25 - 0 - 8$

$= 99$

Partitions

- Ordered Partitions
- Permutations with Repetitions
- Unordered Partitions
- Number of Partitions

Ordered Partitions

- Let A contain n elements and let $n_1, n_2, \ldots n_k$ be positive integers whose sum is n, such that $n_1 + n_2 + \ldots + n_k = n$. Then there exist

$$\frac{n!}{n_1! n_2! \ldots n_k!}$$

different ordered partitions of A of the form $[A_1, A_2, \ldots, A_k]$ where A_1 contains n_1 elements, A_2 contains n_2 elements, …, and A_k contains n_k elements.

Example

- A puzzle has 3 squares, 2 triangles, and 4 circles. How many patterns can be formed by laying these 9 shapes out in a row?
- Answer

$$\frac{9!}{3!2!4!} = 1260$$

Permutations with Repetitions

- Permutations with repetition, or a set of elements in which some of the elements are alike can be expressed as:

$$P(n; n_1, n_2, \ldots n_k) = \frac{n!}{n_1! n_2! \ldots n_k!}$$

where $P(n; n_1, n_2, \ldots n_k)$ denoted the number of permutations of n objects of which n_1 are alike, n_2 are alike, … , n_k are alike.

Example

- Find the number of all possible five-letter words using the letters from the word "apple".
- In five letter, two of them are repeated. So there are totally 5!/2! = 60 ways

Unordered Partitions

- Example: Find the number of way that 12 students can be partitioned into three teams, A_1, A_2 and A_3, so that each team contains four students.
- Answer:
  - Each partition $\{A_1, A_2, A_3\}$ of the students can be arranged in 3!=6 ways as an ordered partition.
  - There are 12!/(4!4!4!) = 34650 ordered partitions.
  - Thus there are 34650/6 = 5775 unordered partitions.

Number of Partitions

- The number of ways to partition a set with n elements into k blocks is given by S(n, k), where S(n, k) satisfies the following recurrence relation:
  - S(n, 1)=1 and S(n, n)=1, for all n≥1.
  - S(n+1, k+1)=(k+1)S(n, k+1)+S(n, k), for n≥1 and 1 ≤ k ≤ n.

Example

- In how many ways can a set with five elements be partitioned into three blocks?
- S(5, 3) = 3S(4, 3) + S(4, 2)
  S(4, 3) = 3S(3, 3) + S(3, 2)
  S(4, 2) = 2S(3, 2) + S(3, 1)
  S(3, 2) = 2S(2, 2) + S(2, 1)
  S(2, 1) = 1
  S(3, 1) = 1
  S(3, 3) = 1
  S(2, 2) = 1
  Therefore, S(5, 3) = 25