Relations and Functions

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Order Pair

- An Ordered Pair consists of two elements, say *a* and *b*, in which one of them, saying *a* is designated as the first element and the other as the second element.
  - The ordered pairs (1,2) and (2,1) are different
  - The set {1, 2} is not an ordered pair
  - Ordered pair can have same elements (1,1)

Cartesian Product

- If X and Y are sets, we let $X \times Y$ denote the set of all order pairs (x, y) where $x \in X$ and $y \in Y$.
- We call $X \times Y$ the **Cartesian Product** of X and Y.
- If $|X| = m$, $|Y| = n$, then $|X \times Y| = mn$
- If A = {1, 2, 3} and B = {a, b}.

  The Cartesian Product $A \times B$

  = {(1, a), (1, b), (2, a), (2, b), (3, a), (3,b) }

Notation for Number System

- $\varnothing$ denotes Null Set
- ***Z*** denotes the set of Integer
- ***Q*** denotes the set of Rational Numbers
- $\boldsymbol{Q}^c$ denotes the set of Irrational Numbers
- ***N*** denotes the set of Natural Numbers
- ***R*** denotes the set of Real Numbers

Introduction to Relations

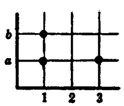
- A **Binary Relation** between sets A and B is a subset of $A \times B$. In other word, a binary relation is a collection of ordered pairs from $A \times B$.
- If A and B are equal, this relation is called **Relation on the Set A**.
- Since relation R is a subset of $A \times B$, any relation R has a complementary relation $\overline{R}$, which is the complement of the set R relative $A \times B$.
- (Do Ex. 1 & 2)

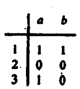
Domain and Range

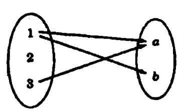
- The **Domain** of a relation R is the set of all first elements of the ordered pair which belong to R, and the **Range** of R is the set of second elements.
- A relation is also written as aRb.
- Example:
  - If A = {1, 2, 3}, B = {a, b, c}, and R = {(1, b), (1, c), (3, b)}. For this relation 1Rb, 1Rc, 3Rb, domain = {1, 3}, range = {b, c}.

Pictorial Representation of Relations

- Let R be a relation from A = {1, 2, 3} to B = {a, b} where R = {(1, a), (1, b), (3, a)}. The represented as follows:



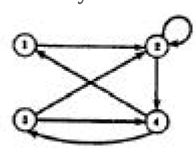




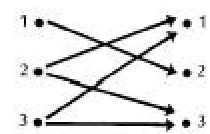
- (Do Ex. 3)

Graphical Representation of Relations

- **Directed Graph** is a way of picturing a relation when it is form a finite set to itself is to write down the elements of the set and then draw an arrow from an elements x to an element y whenever x is related to y.



Directed Graph



Graphical representation of a relation

- (Do Ex. 4 – 7)

Matrix Representation of a Relation

- Let A be a set with *n* elements, and let B be a set with *m* elements and R be a relation between A and B.
  - A = $\{a_1, a_2, \ldots, a_n\}$
  - B = $\{b_1, b_2, \ldots, b_m\}$
- Matrix M is called the **Logical Matrix** for R if

$$M(i, j) = \begin{cases} True & \forall (a_i, b_j) \in R \\ False & \forall (a_i, b_j) \notin R \end{cases}$$

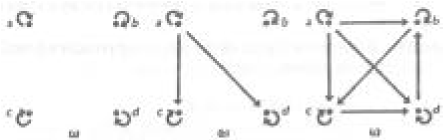
- (Do Ex. 8)

Properties of Relations

- Reflexive
- Symmetric
- Transitive
- Irreflexive
- Antisymmetric

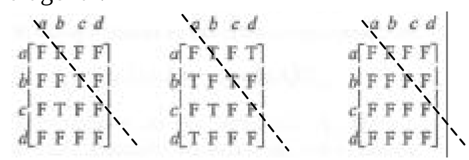
Reflexive

- Let R be a subset of A × A. Then R is called a **Reflexive Relation** if ∀x∈ A, (a, a)∈ R.
- The direct graph of every reflexive relation includes an arrow from every point to the point itself.



Symmetric

- Let R be a subset of A × A. Then R is called a **Symmetric Relation** if (a, b)∈ R ⇒ (b, a)∈ R.
- The matrix representation for the symmetric relations are symmetric with respect to the main diagonal.



Transitive

- A relation R in a set A is called a **Transitive Relation** if $((a, b) \in R \cap (b, c) \in R) \Rightarrow (a, c) \in R$.
- Example:
  - Let W = {a, b, c}, and let R = {(a, b), (c, b), (b, a), (a, c)}.
  - Then R is not a transitive relation because (c, b) ∈ R and (b, a) ∈ R but (c, a) ∉ R.

Irreflexive

- A Relation R on a set S is **Irreflexive Relation** if $x \overline{R} x, \forall x \in R$.
- Example:
  - The relation on the set {a, b, c} given by the set of order pairs {(a, b), (b, c), (c, a)} is irreflexive, because it does not contain any of the ordered pairs (a, a), (b, b) and (c, c).

Antisymmetric

- A Relation R on a set S is **Antisymmetric** if $\forall x, y \in S, (xRy \cap yRx) \Rightarrow (x = y)$.
- Example:
  - The relation "Greater than or equal to" on the set of integer is antisymmetric because if $x, y \in Z$, then $(x \geq y \text{ and } y \leq x) \Rightarrow (x = y)$.

Types of Relations

- Equivalence of Relations
- Partially Ordered Relations
- Universal Relations
- Empty Relations
- Inverse Relations
- Composite Relations

Equivalence Relations

- A relation is an **Equivalence Relation** if it is reflexive, symmetric and transitive and is denoted as ~.

Example

- Provide that the relation = of equality on any set S is an equivalence relation.
  - (1) a = a for every a in S; (Reflexive property)
  - (2) if a = b, then b = a; (Symmetric property)
  - (3) if a = b and b = c, then a = c. (Transitive property)
  - Therefore, S is an equivalent relation.

Partially Ordered Relations

- A Relation on a set is reflexive, antisymmetric and transitive is called **Partially Ordered Relation** on the set.

Example

- If a and b are positive integers, a|b means that a is a divisor of b, i.e. b = ac for some integer c. Show that "|" is a partial ordering of the set of positive integers.

Answer

- By definition, the a|b means that the number b/a is an integer. We need to verify reflexivity, antisymmetry, and transitivity.
  - Reflexivity: $\forall n \in \boldsymbol{Z}^+$, n|n as n/n is 1
  - Antisymmetry: If n|m and m|n, then m/n and n/m are both integers. Since $n/m = (m/n)^{-1}$, the integer n/m has the property that its reciprocal is also an integer. The only such positive integer is 1, and so n/m = 1, i.e. n = m.
  - Transitivity: If n|m and m|p, then p/n = (n/m) x (m/n) is an integer, since it is the product of two other integers.
- It follows that "|" is a partial ordering.

Universal and Empty Relations

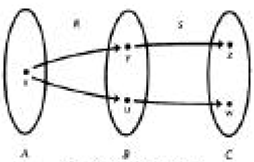
- Universal Relation
  - Let A be any set, then $A \times A$ is known as the **Universal Relation**.
- Empty Relation
  - Let A be any set, then $\varnothing$ is called the **Empty Relation**.

Inverse Relations

- For every relation R between sets A and B is a subset of $A \times B$, the **Inverse Relation** of R is the reverses of the roles of A and B to obtain a relation between B and A.
- The inverse Relation of R is denoted as R^{-1} and the relation between B and A given by
  $R^{-1} = \{(y, x) \mid (x, y) \in R\}$
- (Do Ex. 9 – 10)

Composite Relations

- Let R be a relation between sets A and B, and let S be a relation between B and C. The composition of R and S is the relation between A and C.
- The composite relation of S and R is denoted as S o R, given by S o R = $\{(x, z) \mid x \in A, z \in C, \exists y \in B, xRy \cap ySz\}$
- (Do Ex. 11)



Composite relation $S \circ R$

Introduction to Functions

- A Function is an association of exactly one object from one set (the range) with each object from another set (the domain).
- This means that there must be at least one arrow leaving each point in the domain, and further that there can be no more than one arrow leaving each point in the domain.
- (Do Ex. 12)

Elements of a Function

- Functions are often referred to as **Mappings** or **Transformations**.
- The unique element y = *f* (x) of B assigned to x∈ A by *f* is called the **Image** of x under *f*.
- *f*: A→B indicate *f* is a function from A to B. The set A is called the **Domain** of *f*, and set B is called **Codomain** of *f*.
- The range of *f* denoted by *f* [A], is the set of all images:

  f [A] = {*f*(x) | x ∈ A}
- The **Pre-image** or Inverse Image of a set B contained in the range of *f* is denoted by f^{-1}(B) and is the subset of the domain whose members have images in b.

Example

- The geometric ~~mean~~ function gmean: N x N → R^+ is defined by gmean(x,y) = √(xy)
  - What is the domain of gmean?
  - Explain why the range of gmean is different from the codmain
  - Is gmean 1 one-to-one function? Why?
- Answer
  - The domain of gmean (N,N)
  - The range of gmean is different from the codmain:
    - gmean(x,y) – domain
    - √(xy) – codmain
  - gmean is not a one-to-one function since gmean(1,4) and gmean(2,2) = 2

Graphing Functions

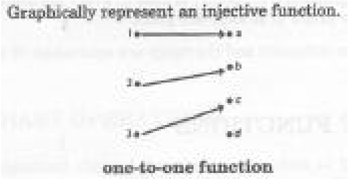
- The set of all ordered pairs of the function *f* plotted in a Cartesian coordinate system is called the **Graph of *f***.
- The graph of a function f is equivalent to the graph of the equation y = *f* (x) as described in algebra.
- (Do Ex. 13 – 14)

Types of Functions

- Injections
- Surjections
- Bijections

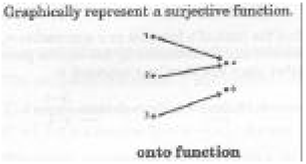
Injection

- Let *f*: A→B be a function. The function *f* is called an **Injective Function**, or an **Injection** if $\forall x, y \in A$, $f(x) = f(y) \Rightarrow x = y$.
- An injective function is also called a **One-to-one** or **1-1** Function.



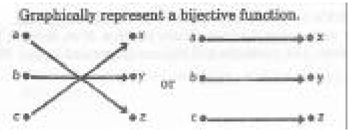
Surjection

- Let *f*: A→B be a function. The function *f* is called a **Surjective Function**, or a **Surjection** if $\exists x \in A$, $\forall y \in B$, $f(x) = y$.
- A surjective function is also called an **Onto** Function.



Bijection

- If a function is both 1-1 and Onto, it is called a **Bijective Function**, or a **Bijection**.
- A bijection from a set A to itself is called a Permutation of the set A.



Example

- Use counter example, show the function f: Z → Z is defined by the rule $f(x) = 4x^2 - 1$ for $x \in Z$ is not bijective function for x is any integers.
- Answer
  - $f(x) = 4x^2 - 1$ is not a 1-to-1 function since (-1, 3) and (1,3) exist

Example

- Give f: Z → Z, Show **$f(x) = (x^2 + 1)/2$** whether it is or not
  - 1 to 1 Function
  - Onto Function
- Answer
  - f(x) is not 1 to 1 function since (-1, 2) and (1,2) ∈ f(x).
  - f(x) is not onto function since –5 in domain B but no x in domain B.
    - $-5 = (x^2+1)/2 \Rightarrow -11 = x^2 \Rightarrow$ impossible.

Limits

- The function ***f*** **(x) approaches the limit L as approaches +¥** if the values of *f* (x) get arbitrarily close to L as x gets arbitrarily large, written as

$$\lim_{x \to +\infty} f(x) = L$$

- (Do Ex. 15 – 16)

Binary Operations

- A **Binary Operation** on a set A is a function *op*: $A \times A \to A$. Thus, a binary operation takes two elements of A and maps them to a third element of A.
- The Binary Operation is denoted as *op*(a, b) or a *op* b, a, b $\in$ A.
- op(a, b) is called **Prefix Notation**.
- a op b is called **Infix Notation**.

Operations of Functions

- Equal Functions
- Sum of Functions
- Difference of Functions
- Product of Functions
- Quotient of Functions
- Composite Functions
- Invertible Functions

Equal Functions

- Two functions f and g are said to be equal if they have the same domain and codomain, and for all x in the domain, $f(x) = g(x)$.
- Example
  - Let $f(x) = (6x - 4)/2$ and $g(x) = 3x - 2$.
  - Then $f = g$, since they both have the same domain and codomain, and for all x in the domain $f(x) = g(x)$.

Sum and Difference of Functions

- Sum of Functions
  - The **Sum of *f* and g**, $f + g$ is defined by
    $(f + g)(x) = f(x) + g(x)$
- Difference of Functions
  - The **Difference of *f* and g**, $f - g$ is defined by
    $(f - g)(x) = f(x) - g(x)$

Product and Quotient of Functions

- Product of Functions
  - The **Product of *f* and g**, $f\,g$ is defined by
    $(f\,g)(x) = f(x) \bullet g(x)$
- Quotient of Functions
  - The **Sum of *f* and g**, f / g is defined by
    $(f / g)(x) = f(x) / g(x)$

Composite Functions

- Since functions are subset of relation, we can form the composition of two function into a **Composite Function**.
- The composition of two functions f and g relates an element a to an element c if there is some element b such that b = f(a) and c = g(b).
- Given two functions *f* and g, the composite function, denoted by *f* o g is defined by
  (f o g)(x) = f(g(x))

Example

- Let f(x) = 3x + 5 and g(x) = 4x – 3, find (f o g)(x)
- (f o g)(x)
- = f (g(x))
- = f (4x-3)
- = 3(4x – 3) + 5
- =12x –9 +5
- = 12x – 4

Invertible Functions

- If the inverse relation of a function is a function, then the function is **Invertible**.
- Let f: A → B be a function. The function f is invertible if and only if f is a bijection.

Example

- Find the inverse of f(x) = 4x – 1

First we solve $y = 4x - 1$ for x in terms of y:

$$y + 1 = 4x \quad \text{or} \quad 4x = y + 1 \quad \text{or} \quad x = \frac{y+1}{4}$$

Now replace x by y and y by x, obtaining $y = (x+1)/4$

Therefore, $f^{-1}(x) = (x+1)/4$

Difference between Function and Relation

- In a function, no two distinct ordered pairs have the same first element.