Question 1 (Compulsory)

(a) Draw a Venn diagram for sets A, B and C that satisfy the following conditions:

(i) $A \subset B$, $C \subset B$, and $A \cap C = \emptyset$. [2 marks]

(ii) $C \subset A$, $B \cap A \neq \emptyset$, and $B \cap C = \emptyset$. [2 marks]

(b) Which of the following conditional statements are true and which are false? Justify your answers.

(i) If $1 + 1 = 2$, then $2 + 2 = 5$. [1 mark]

(ii) If $1 + 1 = 3$, then $2 + 2 = 4$. [1 mark]

(iii) If $1 + 1 = 3$, then $2 + 2 = 5$. [1 mark]

(iv) If $1 + 1 = 2$, then pigs can fly. [1 mark]

(v) If $2 + 2 = 4$, then $1 + 2 = 3$. [1 mark]

(c) Let A be a 3×4 matrix, B be a 4×5 matrix, and C be a 4×4 matrix. Determine which of the following products are defined and find the size of those that are defined.

(i) AB

(ii) BA

(iii) CA

(iv) AA

[5 marks]

(d) Use mathematical induction to prove that

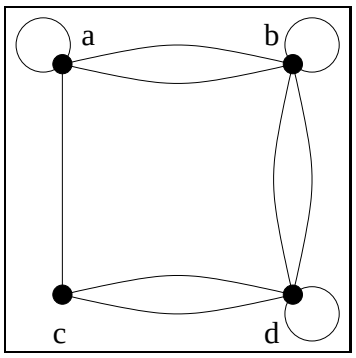
$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

whenever n is a nonnegative integer. [5 marks]

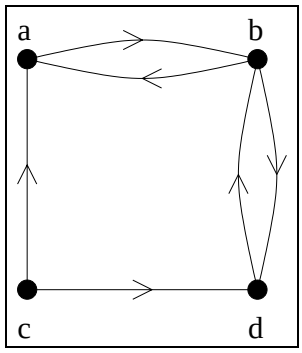
(Question 1 continues on next page)

(Question 1 continued from previous page)

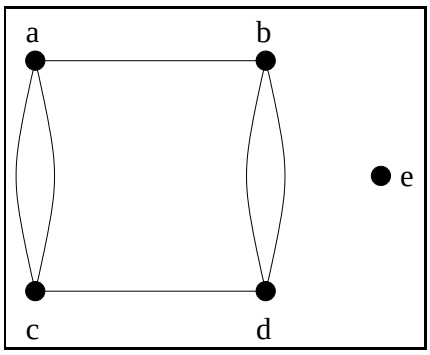
(e) Consider the three graphs A, B and C shown below.



A



B



C

(i) Which graph is disconnected? [1 mark]

(ii) Which graph is a directed multigraph? [1 mark]

(iii) Which graph is a pseudograph that is connected and not directed? [1 mark]

Justify your answers.

(Question 1 continues on next page)

(Question 1 continued from previous page)

(f) There are 325 computer science students at a school. How many ways are there to pick 3 representatives from these students? [2 marks]

(g) Evaluate

$$\lim_{x \to \infty} \frac{3x^7 + 5x^4 - 3x + 4}{9x^7 - x^5}$$

[2 marks]

(h) Two fair dice are rolled. Calculate the probablity of:

(i) both dice showing 6; [1 mark]

(ii) neither dice showing 6; [1 mark]

(iii) at least one of the dice showing 6. [2 marks]

Question 2

(a) Let $A = \{a, b, c\}$, $B = \{b, c, d\}$ and $C = \{b, c, e\}$.

Write down the elements that are contained in the following sets.

(i) $A \cup (B \cap C)$

(ii) $(A \cup B) \cap C$

(iii) $(A \cup B) \cap (A \cup C)$

(iv) $(A - B) - C$

(v) $A - (B - C)$

Are any of these sets equal? [7 marks]

(b) Recall that $P(A)$ denotes the set of all subsets of A.

(i) Write down $P(\{a, b, c\})$. [3 marks]

(ii) A function $F : P(\{a, b, c\}) \to \{0, 1, 2, 3\}$ is defined by

$F(X) =$ the number of elements in X

Write down $F(X)$ for each $X \in P(\{a, b, c\})$. [5 marks]

Question 3

K_n denotes the complete graph on n vertices, and C_n denotes a cycle having n vertices.

(a) (i) Draw K_5 [2 marks]

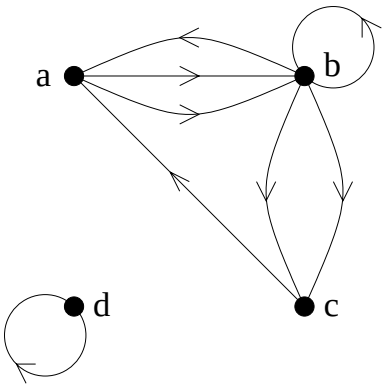
(ii) Draw C_5 [2 marks]

(b) How many edges do the following graphs have?

(i) K_n [2 marks]

(ii) C_n [2 marks]

(c) Consider the directed multigraph shown below.



(i) List the vertices.

(ii) List the edges.

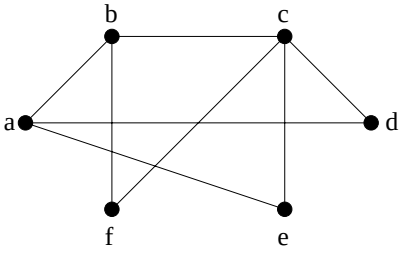
(iii) Find the in–degree and out–degree of each vertex.

[4 marks]

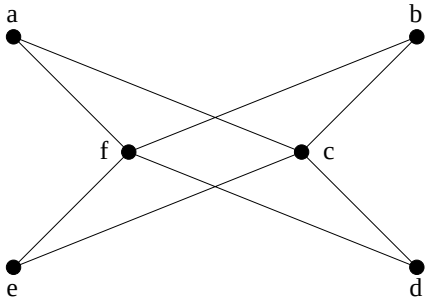
(Question 3 continues on next page)

(Question 3 continued from previous page)

(d) Consider the two graphs below. One graph is bipartite, the other is not bipartite. State, giving reasons, which graph is bipartite. For this bipartite graph find two sets of vertices A and B so that vertices within the same set are nonadjacent. [3 marks]



A



B

Question 4

(a) Use De Morgan's laws to write negations for the following statements.

(i) Chee Leong is a multimedia major and Chee Leong's sister is a computer science major. [2 marks]

(ii) The power is switched off or the machine is unplugged. [2 marks]

(b) Construct truth tables for the following statements.

(i) $(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$ [3 marks]

(ii) $(p \wedge \sim q) \wedge (\sim p \vee q)$ [3 marks]

Which statement is a tautology and which is a contradiction? Justify your answer. [2 marks]

(c) Disprove the following statements by giving a counterexample.

(i) The product of any two prime numbers is prime. [1 mark]

(ii) The difference of any two odd integers is odd. [1 mark]

(iii) For all integers n, $n^2 - n + 11$ is a prime number. [1 mark]

Question 5

(a) What is the minimum number of students needed to guarantee that at least six students will receive the same grade, if there are five possible grades A, B, C, D and F? Justify your answer. [4 marks]

(b) A pair of fair dice are rolled. Let X be the sum of the numbers that appear. Let S be the set of values that X can take.

(i) Write down the elements of S. [2 marks]

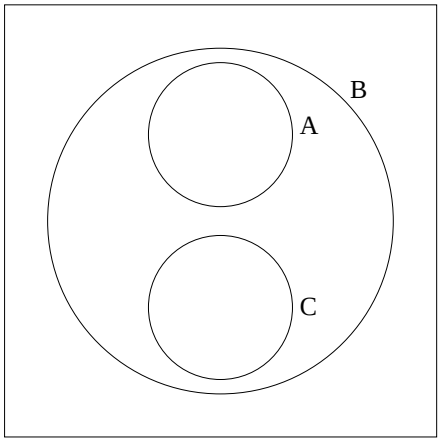
(ii) Write down the distribution of X. [5 marks]

(iii) Calculate the expected value of X. [4 marks]

Answer 1

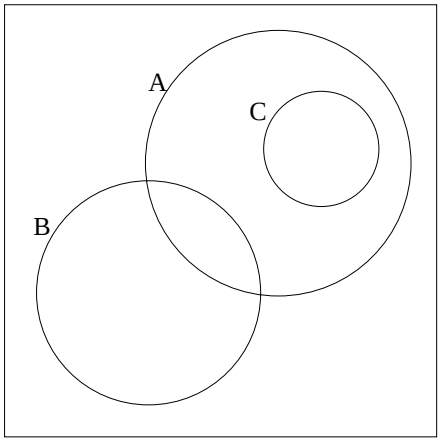
Please do not award half marks.

(a) (i) $A \subset B$, $C \subset B$, $A \cap C = \emptyset$.



Two marks. Deduct one mark for each error up to a maximum of two. [2 marks]

(ii) $C \subset A$, $B \cap A \neq \emptyset$, $B \cap C = \emptyset$.



Two marks. Deduct one mark for each error up to a maximum of two. [2 marks]

(Answer 1 continues on next page)

(Answer 1 continued from previous page)

(b) (i) $1 + 1 = 2$ is true, $2 + 2 = 5$ is false, so the implication is false. [1 mark]

(ii) $1 + 1 = 3$ is false, $2 + 2 = 4$ is true, so the implication is true. [1 mark]

(iii) $1 + 1 = 3$ is false, $2 + 2 = 5$ is false, so the implication is true. [1 mark]

(iv) $1 + 1 = 2$ is true, pigs can fly is false, so the implication is false. [1 mark]

(v) $2 + 2 = 4$ is true, $1 + 2 = 3$ is true, so the implication is true. [1 mark]

Only award marks if the correct reason is given

(c) (i) AB is defined, size 3×5. *One mark for stating AB is defined, one mark for size of AB.*

(ii) BA is not defined. *One mark.*

(iii) CA is not defined. *One mark.*

(iv) AA is not defined. *One mark.*

[5 marks]

(d) Step 1: show that statement is true for $n = 1$. When $n = 1$, LHS=1, RHS=1. *Two marks.*

Step 2: Prove that if it is true for n then it is true for $n + 1$.

$$\begin{aligned} \sum_{j=1}^{n+1} j &= n + 1 + \sum_{j=1}^{n} j && \textit{One mark.} \\ &= n + 1 + \frac{n(n+1)}{2} && \textit{One mark.} \\ &= \frac{(n+1)(n+2)}{2} && \textit{One mark.} \end{aligned}$$

[5 marks]

(Answer 1 continues on next page)

(Answer 1 continued from previous page)

(e) (i) C is disconnected because there is no path from a to e. [1 mark]

(ii) B is a directed multigraph because it is a multigraph with edges that have a direction indicated. [1 mark]

(iii) A is a pseudograph as pseudographs may have any number of edges between vertices (including loops) and it is connected and not directed. [1 mark]
Only award marks if both the graph and the reason is correct.

(f) ${}^{325}C_3 = 5668650$. *One mark for method, one mark for correct answer.* [2 marks]

(g)

$$\lim_{x\to\infty} \frac{3x^7 + 5x^4 - 3x + 4}{9x^7 - x^5} = \lim_{x\to\infty} \frac{3 + 5x^{-3} + 3x^{-6} + 4x^{-7}}{9 - x^{-2}} \quad \textit{One mark.}$$
$$= \frac{1}{3} \quad \textit{One mark.}$$

[2 marks]

(h) (i) $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ [1 mark]

(ii) $\frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$ [1 mark]

(iii) $1 - P(\text{neither dice showing } 6) = \frac{11}{36}$ [2 marks]

Answer 2

Please do not award half marks.

(a) (i) $A \cup (B \cap C) = \{a, b, c\}$ *One mark*

(ii) $(A \cup B) \cap C = \{b, c\}$ *One mark*

(iii) $(A \cup B) \cap (A \cup C) = \{a, b, c\}$ *One mark*

(iv) $(A - B) - C = \{a\}$ *One mark*

(v) $A - (B - C) = \{a, b, c\}$ *One mark*

$A \cup (B \cap C)$, $(A \cup B) \cap (A \cup C)$ and $A - (B - C)$ are equal.
Two marks. Subtract one mark for every error or omission up to a maximum of two. [7 marks]

(b) (i) $P(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
Three marks. Subtract one mark for every error or omission up to a maximum of three. [3 marks]

(ii)

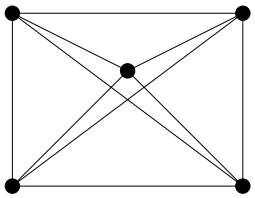
$$\begin{aligned} F(\emptyset) &= 0 \\ F(\{a\}) &= 1 \\ F(\{b\}) &= 1 \\ F(\{c\}) &= 1 \\ F(\{a, b\}) &= 2 \\ F(\{a, c\}) &= 2 \\ F(\{b, c\}) &= 2 \\ F(\{a, b, c\}) &= 3 \end{aligned}$$

Five marks. Subtract one mark for every error or omission up to a maximum of five. [5 marks]

Answer 3

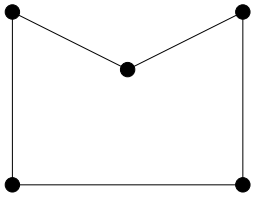
Please do not award half marks.

(a) (i) K_5:



Two marks. Subtract one mark for every error up to a maximum of two. [2 marks]

(ii) C_5:



Two marks. Subtract one mark for every error up to a maximum of two. [2 marks]

(b) (i) K_n has n vertices. Each of these n vertices has $n-1$ edges leaving it. Each edge is attached to 2 vertices and so there are $n(n-1)/2$ edges. *One mark for correct answer, one mark for justification.* [2 marks]

(ii) C_n has n verices with 2 edges leaving each of these n vertices. Each edge is attached to 2 vertices and so there are n edges *One mark for correct answer, one mark for justification.*. [2 marks]

(c) (i) a, b, c, d. *One mark.*

(ii) $ab, ab, ba, bb, bc, bc, dd, ca$. *One mark.*

(iii) In degree: $a = 2$, $b = 3$, $c = 2$, $d = 1$. *One mark.*
Out degree: $a = 2$, $b = 4$, $c = 1$, $d = 1$. *One mark.*

[4 marks]

(d) Graph B is bipartite. *One mark.* $A = \{c, f\}$ and $B = \{a, b, d, e\}$. *Two marks. Only award the mark for saying B is bipartite if the candidate has shown an understanding of what a bipartite graph is.* [3 marks]

Answer 4

Please do not award half marks.

(a) (i) Chee Leong is *not* a multimedia major *or* Chee Leong's sister is *not* a computer science major. *One mark for* or *and one mark for both* nots. [2 marks]

(ii) The power is *not* switched off *and* the machine is *not* unplugged. *One mark for* and *and one mark for both* nots. [2 marks]

(b) (i) $(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge \sim q$	$\sim p \vee (p \wedge \sim q)$	$(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	F	T	T
F	F	T	T	F	F	T	T

Three marks. Subtract one mark for each error up to a maximum of three. [3 marks]

(ii) $(p \wedge \sim q) \wedge (\sim p \vee q)$

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \vee q$	$(p \wedge \sim q) \wedge (\sim p \vee q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

Three marks. Subtract one mark for each error up to a maximum of three. [3 marks]

(i) is a tautology as it is always true. *One mark.* (ii) is a contradiction because it is always false. *One mark. Do not award any marks unless both the correct answer and reason are given.* [2 marks]

(c) *Award marks for any suitable contradiction.*

(i) E.g. 3 and 5 are both prime numbers, but $3 \times 5 = 15$ is not prime. [1 mark]

(ii) E.g. 5 and 3 are odd integers, but $5 - 3 = 2$ is even. [1 mark]

(iii) E.g. When $n = 11$, $n^2 - n + 11 = 121$ is not prime. [1 mark]

Answer 5

Please do not award half marks.

(a) If there are 5 grades available then by the generalised pigeon hole principle the minimum number of students needed to guarantee that at least n students receive the same grade is the smallest integer N such that $N = 5(n-1)+1$. In this case, $n = 6$, so $N = 26$. *One mark for the correct answer, one mark for mentioning generalised pigeon hole principle, two marks for the method.* [4 marks]

(b) (i) $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. *Two marks. Deduct one mark for each error up to a maximum of two.* [2 marks]

(ii) Distribution of X:

s	$P(X = s)$
2	1/36
3	1/18
4	1/12
5	1/9
6	5/36
7	1/6
8	5/36
9	1/9
10	1/12
11	1/18
12	1/36

Five marks. Deduct one mark for each error up to a maximum of five. [5 marks]

(iii) $E(X) = \sum_{s=2}^{12} sP(X = s)$=7. *One mark for correct answer, three marks for method.* [4 marks]