Question 1 (Compulsory)

(a) For each of the following sets, determine whether 2 is an element of that set.

(i) $\{x \mid x \text{ is an integer greater than } 1\}$

(ii) $\{x \mid x \text{ is the square of an integer}\}$

(iii) $\{2, \{2\}\}$

(iv) $\{\{2\}, \{\{2\}\}\}$

(v) $\{\{2\}, \{2, \{2\}\}\}$

(vi) $\{\{\{2\}\}\}$

[3 marks]

(b) The matrices A and B are given by

$$A = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 4 \\ 0 & 5 \end{pmatrix}$$

(i) Calculate $(AB)^{\mathrm{T}}$ [2 marks]

(ii) Calculate $B^{\mathrm{T}}A^{\mathrm{T}}$ [2 marks]

(c) Let p and q be the propositions

p: It is below freezing
q: It is snowing

Write the following propositions using p and q and logical connectives.

(i) It is below freezing and snowing. [1 mark]

(ii) It is below freezing but not snowing. [1 mark]

(iii) It is not below freezing and it is not snowing. [1 mark]

(iv) It is either below freezing or it is snowing, and it is not snowing if it is below freezing. [1 mark]

(v) It is below freezing precisely when it is snowing. [1 mark]

(d) A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0? [4 marks]

(Question 1 continues on next page)

(Question 1 continued from previous page)

(e) Use mathematical induction to prove that

$$3\sum_{k=0}^{n} 5^k = 3\left(5^{n+1} - 1\right)/4$$

whenever n is a nonnegative integer. [6 marks]

(f) Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation

$$R = \{(a, b) \mid a, b \in A, a \text{ divides } b\}$$

[2 marks]

(g) How many ways are there to choose 6 items from 10 distinct items when

(i) the items in the choices are ordered and repetition is not allowed? [2 marks]

(ii) the items in the choices are ordered and repetition is allowed? [2 marks]

(iii) the items in the choices are unordered and repetition is not allowed? [2 marks]

Question 2

(a) Consider the following relations on the set $\{1, 2, 3, 4\}$

$$\begin{aligned}
R_1 &= \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\} \\
R_2 &= \{(1,1),(1,2),(2,1)\} \\
R_3 &= \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\} \\
R_4 &= \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\} \\
R_5 &= \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\} \\
R_6 &= \{(3,4)\}
\end{aligned}$$

(i) Which of these relations are reflexive? Explain. [2 marks]

(ii) Which of these relations are symmetric? Explain. [2 marks]

(iii) Which of these relations are antisymmetric? Explain. [2 marks]

(iv) Which of these relations are transitive? Explain. [2 marks]

(b) Let A, B and C be sets. Show, using algebraic laws, that

(i) $(A \cup B) \subseteq (A \cup B \cup C)$ [2 marks]

(ii) $(A \cap B \cap C) \subseteq (A \cap B)$ [2 marks]

(c) Why is f not a function from $\mathbf{R}$ to $\mathbf{R}$ in the following equations?

(i) $f(x) = 1/x$ [1 mark]

(ii) $f(x) = \sqrt{x}$ [1 mark]

(iii) $f(x) = \pm\sqrt{x^2 + 1}$ [1 mark]

Question 3

(a) (i) Suppose $n = 1 \pmod 3$. By writing $n = 3k + 1$, where k is an integer, show that $n^2 = 1 \pmod 3$. [2 marks]

(ii) Suppose $n = 2 \pmod 3$. Calculate $n^2 \pmod 3$. [2 marks]

(iii) Use (i) and (ii) to prove the implication "if n is an integer not divisible by 3, then $n^2 = 1 \pmod 3$". [3 marks]

(b) Show that $[\sim p \wedge (p \vee q)] \rightarrow q$ is a tautology by constructing a truth table. [3 marks]

(c) Using algebraic laws show that $\sim ((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$ [5 marks]

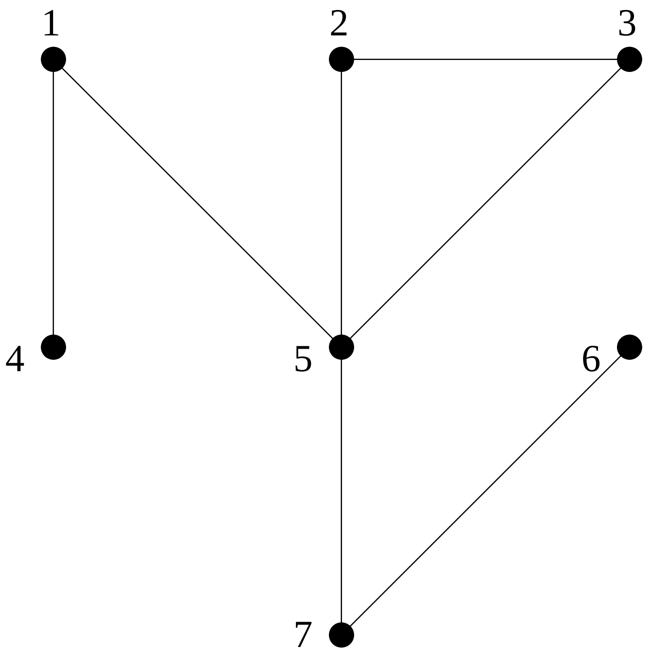
Question 4

(a) Show that every tree with at least two vertices has at least one vertex of odd degree. [3 marks]

(b) The following is an adjacency matrix of a graph. Draw the graph. [3 marks]

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

(c) Consider the graph shown below.

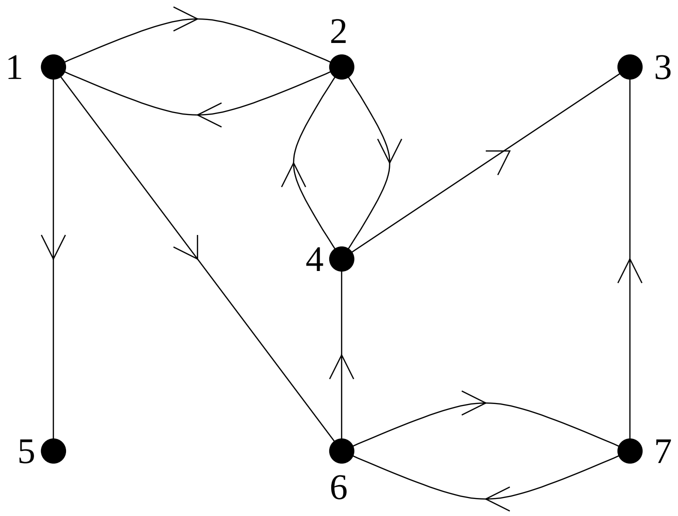


(Question 4 continues on next page)

(i) Write down the distance matrix. [2 marks]

(ii) Define a walk, a trail and an Eulerian trail of a graph G. Find an Eulerian trail in the given graph. [4 marks]

(d) Consider the directed graph shown below.



Write down the adjacency matrix for this graph. [3 marks]

Question 5

(a) How many bit strings of length eight either start with a 1 bit or end with the two bits 00? [4 marks]

(b) Assume that in a group of six people, any pair of individuals consists of either two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group. [5 marks]

(Question 5 continues on next page)

(Question 5 continued from previous page)

(c) A study was carried out to determine the efficiency of three different drugs A, B and C in relieving headache pain. The following results were obtained:

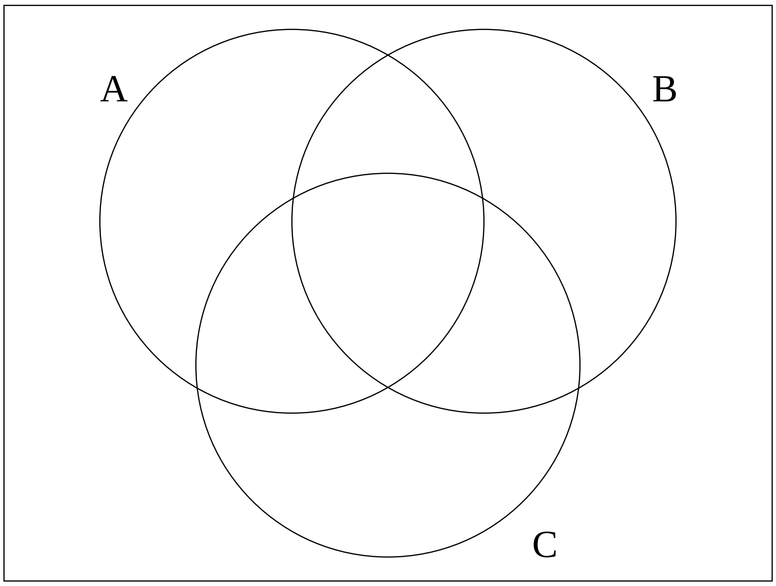
40 subjects were given the chance to use all three drugs
23 reported relief from drug A
18 reported relief from drug B
31 reported relief from drug C
11 reported relief from both drugs A and B
19 reported relief from both drugs A and C
14 reported relief from both drugs B and C
37 reported relief from at least one of the drugs

Note that some of the 23 subjects who reported relief from drug A may have also reported relief from drugs B or C, with similar remarks applying to other data.

(i) How many people got relief from none of the drugs? [1 mark]

(ii) How many people got relief from all three drugs? [2 marks]

(iii) Let A be the set of all subjects who got relief from drug A, B the set of all subjects who got relief from drug B, and C the set of all subjects who got relief from drug C. Make a copy of the diagram shown below and fill in the numbers for all eight regions. [2 marks]



(iv) How many subjects got relief from A only? [1 mark]

Answer 1

Please do not award half-marks.

(a) (i) yes

(ii) no

(iii) yes

(iv) no

(v) no

(vi) no

Award one mark for two or three correct answers, two marks for four or five correct answers, three marks for six correct answers. [3 marks]

(b) (i) $AB = \begin{pmatrix} -4 & 23 \\ -8 & 11 \end{pmatrix}$

$(AB)^{\mathrm{T}} = \begin{pmatrix} -4 & -8 \\ 23 & 11 \end{pmatrix}$

Award one mark for correct multiplication of matrices and one for correctly transposing the matrix. [2 marks]

(ii) Either note that $B^{\mathrm{T}}A^{\mathrm{T}} = (AB)^{\mathrm{T}}$ and use part (i), or

$A^{\mathrm{T}} = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$ $\qquad B^{\mathrm{T}} = \begin{pmatrix} -2 & 0 \\ 4 & 5 \end{pmatrix}$

$B^{\mathrm{T}}A^{\mathrm{T}} = \begin{pmatrix} -4 & -8 \\ 23 & 11 \end{pmatrix}$

Award two marks for first method, for second method award one mark for correct transposes and one mark for correct multiplication. [2 marks]

(c) (i) $p \wedge q$ [1 mark]

(ii) $p \wedge \sim q$ [1 mark]

(iii) $\sim p \wedge \sim q$ [1 mark]

(iv) $(p \vee q) \wedge (p \to \sim q)$ [1 mark]

(v) $q \leftrightarrow p$ [1 mark]

(Answer 1 continues on next page)

(Answer 1 continued from previous page)

(d) Let E be the event that at least one of the 10 bits is 0. Then $\bar{E}$ is the event that all the bits are 1s. Since the sample space S is the set of all bit strings of length 10, it follows that

$$\begin{aligned} P(E) &= 1 - P(\bar{E}) && \textit{One mark} \\ &= 1 - \tfrac{|\bar{E}|}{|S|} && \textit{One mark} \\ &= 1 - \tfrac{1}{2^{10}} && \textit{One mark} \\ &= 1 - \tfrac{1}{1024} \\ &= \tfrac{1023}{1024} && \textit{One mark} \end{aligned}$$

[4 marks]

(e) Let $P(m)$ be the statement $3\sum_{k=0}^{m} 5^k = 3\left(5^{m+1} - 1\right)/4$. *One mark.*
Step 1. Prove $P(0)$.
$3\sum_{k=0}^{0} 5^k = 3$
$3\left(5^{0+1} - 1\right)/4 = 3$ *One mark.*

Step 2. Assume $P(m)$ and prove $P(m+1)$. *One mark.*

$$\begin{aligned} 3\sum_{k=0}^{m+1} 5^k &= 3\sum_{k=0}^{m} 5^k + 3 \times 5^{m+1} && \textit{One mark} \\ &= \tfrac{3\left(5^{m+1}-1\right)}{4} + 3 \times 5^{m+1} && \textit{One mark} \\ &= \tfrac{3\left(5^{m+1}-1+4\times 5^{m+1}\right)}{4} \\ &= \tfrac{3\left(5^{m+2}-1\right)}{4} && \textit{One mark} \end{aligned}$$

[6 marks]

(f) $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
Two marks. Subtract one mark for each mistake up to a maximum of two. [2 marks]

(g) (i) $\frac{n!}{(n-k)!} = \frac{10!}{(10-6)!} = 151,100$ [2 marks]

(ii) $n^k = 10^6 = 1,000,000$ [2 marks]

(iii) $\frac{n!}{(n-k)!k!} = \frac{10!}{(10-6)!6!} = 210$ [2 marks]

For each part award one mark for the correct formula and one mark for the correct answer.

Answer 2

Please do not award half-marks.

(a) (i) The relations R_3 and R_5 are reflexive since they both contain all pairs of the form (a, a), for all elements a. [2 marks]

(ii) The relations R_2 and R_3 are symmetric because in each case (b, a) belongs to the relation whenever (a, b) does. [2 marks]

(iii) The relations R_4, R_5 and R_6 are all antisymmentric because for each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) belong to the relation. [2 marks]

(iv) The relations R_4, R_5 and R_6 are transitive. For each of these relations, if (a, b) and (b, c) belong to the relation, then so does (a, c). [2 marks]

In each case, award one mark for the correct relations and one mark for the correct explanation.

(b) (i)

$A \subseteq (A \cup B \cup C)$	Defn of $\cup$	
$B \subseteq (A \cup B \cup C)$	Defn of $\cup$	*One mark*
$(A \cup B) \subseteq (A \cup B \cup C)$	Defn of $\subseteq$	*One mark*

[2 marks]

(ii)

$(A \cap B \cap C) \subseteq A$	Defn of $\cap$	
$(A \cap B \cap C) \subseteq B$	Defn of $\cap$	*One mark*
$(A \cap B \cap C) \subseteq (A \cap B)$	Defn of $\subseteq$	*One mark*

[2 marks]

(c) (i) $f(0)$ is not defined. [1 mark]

(ii) $f(x)$ is not defined for $x < 0$. [1 mark]

(iii) $f(x)$ is not well–defined since there are two distinct values assigned to each x. [1 mark]

Answer 3

Please do not award half-marks.

(a) (i) $(3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ *One mark.*
It follows that $n^2 = 1 \pmod 3$ *One mark.* [2 marks]

(ii) We may write $n = 3k + 2$ *One mark.*
Then $n^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$, and so it follows that $n^2 = 1 \pmod 3$ *One mark.* [2 marks]

(iii) If n is not divisible by 3 then either $n = 3k + 1$ or $n = 3k + 2$ *One mark.* In each case, $n^2 = 1 \pmod 3$ *One mark.* Hence, "if n is an integer not divisible by 3, then $n^2 = 1 \pmod 3$ *One mark.* [3 marks]

(b) *Award two marks for a completely correct truth table, subtract one mark for each mistake up to a maximum of two. Award one mark for noting all entries in last column are T and so it is a tautology.*

p	q	$\sim p$	$p \vee q$	$\sim p \wedge (p \vee q)$	$[\sim p \wedge (p \vee q)] \to q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

[3 marks]

(c)
$\sim ((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q)$
$\equiv \sim (\sim p \wedge (q \vee \sim q)) \vee (p \wedge q)$ distributive law (*One mark*)
$\equiv \sim (\sim p \wedge t) \vee (p \wedge q)$ negation law for $\vee$ (*One mark*)
$\equiv \sim (\sim p) \vee (p \wedge q)$ identity law for $\wedge$ (*One mark*)
$\equiv p \vee (p \wedge q)$ double negative law (*One mark*)
$\equiv p$ absorption law (*One mark*)

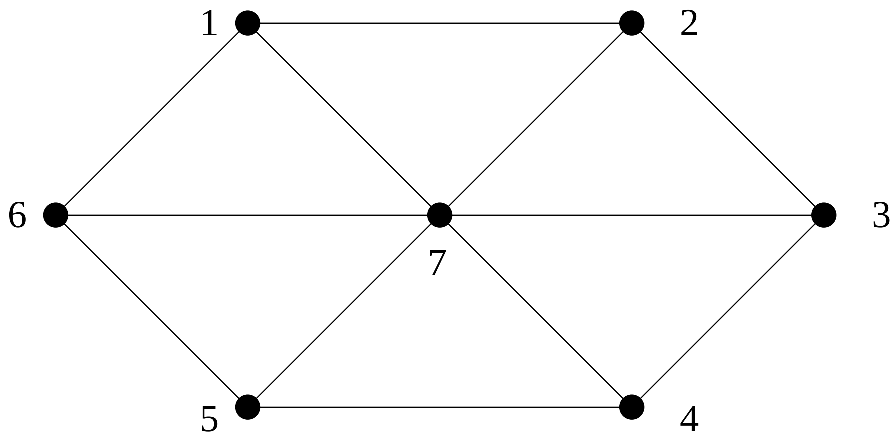
[5 marks]

Answer 4

Please do not award half-marks.

(a) If every vertex were of even degree then since trees are connected (*one mark*) the tree would have an Euler cycle (*one mark*). Since trees have no cycles at all, this cannot happen (*one mark*). [3 marks]

(b)



Three marks, subtract one for each mistake up to a maximum of three. [3 marks]

(c) (i)

$$\begin{pmatrix} 0 & 2 & 2 & 1 & 1 & 3 & 2 \\ 2 & 0 & 1 & 3 & 1 & 3 & 2 \\ 2 & 1 & 0 & 3 & 1 & 3 & 2 \\ 1 & 3 & 3 & 0 & 2 & 4 & 3 \\ 1 & 1 & 1 & 2 & 0 & 2 & 1 \\ 3 & 3 & 3 & 4 & 2 & 0 & 1 \\ 2 & 2 & 2 & 3 & 1 & 1 & 0 \end{pmatrix}$$

Two marks, Subtract one for each mistake up to a maximum of two. [2 marks]

(ii) A walk is a sequence of vertices $v_0, v_1, \ldots, v_k$ of G where $v_i v_{i+1}$ is an edge of G *One mark*.
A trail is a walk in which the edges are distinct *One mark*.
A Eulerian trail is a trail that includes each edge of the graph *One mark*.
Eulerian trail: 41523576 or 41532576 *One mark*. [4 marks]

(Answer 4 continues on next page)

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(d) $$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Three marks, subtract one for each mistake up to a maximum of three. [3 marks]

Answer 5

Please do not award half-marks.

(a) Constructing a bit string of length eight beginning with a 1 bit can be done in $2^7 = 128$ ways *One mark*. Constructing a bit string of length eight ending in 00 can be done in $2^6 = 64$ ways *One mark*. Both tasks can be done in $2^5 = 32$ ways *One mark*.

Consequently, the number of bit strings of length eight that begin with a 1 and end with 00 is given by $128 + 64 - 32 = 160$ *One mark*.

[4 marks]

(b) Let A be one of the six people. Of the other five people in the group, there are either three or more friends of A, or three or more enemies of A (*one mark*) which follows from the generalised pigeonhole principle (*one mark*). In the former case, suppose B, C and D are friends of A. If any two of these three individuals are friends then these two and A form a group of three mutual friends (*one mark*). Otherwise B, C and D form a set of mutual enemies (*one mark*). The proof for the later case follows similarly (*one mark*).

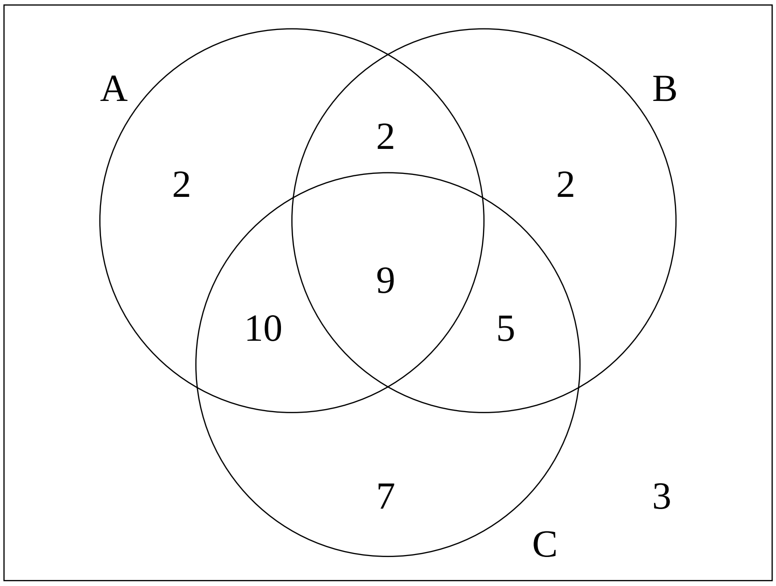
[5 marks]

(Answer 5 continues on next page)

(Answer 5 continued from previous page)

(c) (i) The number of people that got no relief is equal to the number of people in the class minus the number of people that got some relief from at least one of the drugs, or $40 - 37 = 3$ (*one mark*). [1 mark]

(ii) $\mid A \mid + \mid B \mid + \mid C \mid - \mid A \cap B \mid - \mid A \cap C \mid - \mid B \cap C \mid + \mid A \cap B \cap C \mid = \mid A \cup B \cup C \mid$ *one mark*
so the number of people getting relief from all three drugs was 9 (*one mark*). [2 marks]

(iii) 

Two marks, subtract one for each mistake up to a maximum of two. [2 marks]

(iv) 2. [1 mark]